Roll No. .... Total No. of Sections : 4

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Code No. : B03/103

III Semester Examination

M.Sc.

MATHEMATICS

Paper I

[Integration Theory and Functional Analysis (I)]

Time : Three Hours] [Maximum Marks : 80

[Minimum Passing Marks : 16

Note : *Part A and B of each question in each unit consists of Very Short Answer Type Questions which are to be answered in one or two sentences. Part C (Short Answer Type) of each question will be answered 200-250 words. Part D (Long Answer Type) of each question should be answered within the word limit 400-450.*

Unit–I

1. (A) Define positive set. **2**

(B) Define absolute continuous measure. **2**

(C) Prove that the union of countable collection of positive sets is positive. **4**

Or

Let $[(A_i \times B_i)]$ be a countable disjoint collection of measurable rectangles whose union is a measurable rectangle $A \times B$. Then show that

$$\lambda\,(A \times B) = \sum \lambda\,(A_i \times B_i).$$

(D) State and prove Raydon-Nikodym theorem. **12**

Or

State and prove Riesz Representation theorem.

Unit–II

2. (A) Prove that : **2**

$$y - x \quad = \quad x - y \quad .$$

(B) Define quotient space. **2**

(C) Prove that R^n is an *nls* with the following norm : **4**

$$x \quad = \left(\sum_{i=1}^{n} | x_i |^2 \right)^{1/2} .$$

(ii) The converse of (i) is not generally true.

(iii) If dim X < ∞, then weak convergence implies strong convergence.

Or

Prove that C* is isometrically isomorphic to l_1.

Unit–IV

4. (A) Define fixed point. **2**

(B) Define convex function. **2**

(C) Let T be a mapping of complete metric space X into itself. Let T be a contraction on closed ball

$$\overline{B_r(x_0)} = \{x \in X : d(x, x_0) \leq r\}$$

Suppose that :

$$d(x_0 \; Tx_0) < (1 - \alpha) \, r$$

Then show that iterative sequence defined by

$$x_n = T^n{}_{x_0} = Tx_{n-1}$$

converges to an $x \in \overline{B_r(x_0)}$ and this is the unique fixed point of T. **4**

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Or

If T : X → X* is a duality map, then show that it is monotone and coercive.

(D) State and prove Picard theorem. **12**

Or

Define duality map. Show that a duality map can be constructed in any *nls*. Further show that in general, a duality map T : X → X* is multi-valued, and it is single valued if X* is strictly convex.

Or

Let $a \in X$ and $\alpha \in K$, $\alpha \neq 0$. Then show that the mapping

$$x \in X \to \alpha x + a \in X$$

is a homeomorphism of X onto itself.

(D) Let X be normed space. Then show that the mappings :

(a) $(x, y) \in X \times X \to x + y \in X$

(b) $(\lambda, x) \in K \times X \to \lambda x \in X$

(c) $(x, y) \in X \times X \to d(x, y) = \quad y - x \quad \in R$

(d) $n \in X \to \quad x \quad \in R$

are all continuous. **12**

Or

State and prove Riesz lemma. By applying Riesz lemma show that the closed unit ball in a *nls* is compact if and only if the space is finite-dimensional.

Unit–III

3. (A) Define weak convergence. **2**

(B) Define bounded linear transformations. **2**

(C) If T is a linear transformation from a normal linear space X into a normed linear space Y, then show that the following three conditions are equivalent : **4**

(i) T is bounded,

(ii) T is continuous,

(iii) T is continuous at one point.

Or

Let x_n be a weakly convergent sequence in a normed space X, *i.e.*, $x_n \xrightarrow{w} x$. Then show that the sequence (x_n) is bounded.

(D) Let (x_n) be a sequence in a normed linear space X. Then prove that : **12**

(i) Strong convergence implies weak convergence with the same limit.