

** boundary point:- Let (X, d) be a metric space and $S \subseteq X$. A point $x \in X$ is called boundary point of the set S if every ball $B_X(x, r)$ contains at least one point of S and at least one point of $X \setminus S$.

* The set of all boundary points of S is called the boundary of ' S ' and is denoted by ∂S

e.g. ~~the set of all real number i.e~~

e.g. The boundary of set of all rational numbers is the set of all real numbers i.e $\mathbb{R}$.

Some Notations :-

(A) Geometric Notations:-

(i) $\mathbb{R}^n \rightarrow n$ -dimensional Euclidean space and

In particular, for $n=1$, $\mathbb{R}^1 = \mathbb{R}$

(ii) $e_i = (0, 0, \dots, 0, \underset{\downarrow}{1}, 0, \dots, 0)$

$\downarrow$
i-th place

= i-th standard co-ordinate vector.

(iii) A typical point x in $\mathbb{R}^n$ is $x = (x_1, x_2, \dots, x_n)$

Sometimes we will also regard x as a column vector

(iv) $\mathbb{R}_+^n = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$

= open upper half plane

e.g. if $n=2$

$$R_+^2 = \{ (x, y) \in R^2 \mid y > 0 \}$$

(2)

if $n=3$

$$R_+^3 = \{ (x, y, z) \in R^3 \mid z > 0 \}$$

(v) A typical point in R^{n+1} will often be denoted as $(x, t) = (x_1, x_2, \dots, x_n, t)$

and we usually interpret $t = x_{n+1} = \text{time}$

A point $x \in R^n$ will sometimes be written as (x', x_n) ; $x' = (x_1, x_2, \dots, x_{n-1}) \in R^{n-1}$

(vi) U, V , and W are usually open subsets of R^n and $\partial U = \text{boundary of } U$

Also closure of $U = \bar{U} = U \cup \partial U$

(vii) $U_T = U \times (0, T)$

(viii) $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$

$$x \cdot y = \sum_{i=1}^n x_i \cdot y_i$$

$$\text{and } \|x\| = |x| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

= Euclidean norm of x

(ix) $B(x, r) = \text{closed ball in } R^n \text{ with centre at } x \text{ and having radius } r, r > 0$
 $= \{ y \in R^n \mid |x - y| \leq r \}$

(x) $B^0(x, r) = \text{open ball in } R^n \text{ with centre at } x \text{ and radius } r > 0$

$$= \{y \in \mathbb{R}^n \mid |x-y| < 1\}$$

(X.1) $\alpha(n)$ = Volume of Unit ball $B(0,1)$ in $\mathbb{R}^n$

$$= \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

$n \cdot \alpha(n)$ = surface area of unit sphere/Ball

$$= \partial B(0,1) \text{ in } \mathbb{R}^n.$$

(B) Notation for Functions

(4)

(i) Let $U \subseteq \mathbb{R}^n$

If $u: U \rightarrow \mathbb{R}$, we write

$$u(x) = u(x_1, x_2, \dots, x_n), \quad x \in U$$

We say u is smooth provided u is ~~is~~
• infinitely differentiable.

(ii) for $U \subseteq \mathbb{R}^m$

if $u: U \rightarrow \mathbb{R}^m$, we write

$$u(x) = (u^1(x), u^2(x), u^3(x), \dots, u^m(x)); \quad x \in U$$

where $u^i: U \rightarrow \mathbb{R}$ and is called component of u $\forall i$ ($1 \leq i \leq m$)

(iii) Averages :-

$$\int_{B(x;r)} f dy = \frac{1}{\alpha(n)r^n} \int_{B(x;r)} f dy$$

$$= \text{Average of } f \text{ ball } B(x;r)$$

$$\text{and } \int_{\partial B(x;r)} f ds = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x;r)} f ds$$

or surface)

$$= \text{Average of } f \text{ over the sphere } \partial B(x;r)$$

(C) Notation for Derivative

Assume $u: U \rightarrow \mathbb{R}$ for $x \in U$

$$(i) \frac{\partial u}{\partial x_i}(x) = \lim_{h \rightarrow 0} \frac{u(x + he_i) - u(x)}{h}, \quad \text{provided this limit exists.}$$

similarly, $\frac{\partial^2 y}{\partial x_i \partial x_j} = u_{ij} x_j$

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = u_{ijk} x_k$$

(D) Multiindex Notations:-

(i) A vector of the form $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where each component α_i is a non-negative integer is called ~~multi~~ multiindex of order

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

e.g.:- $(0, 0, \dots, 0)$ is a multiindex of order 0.

$(1, 0, \dots, 0), (0, 1, \dots, 0)$ are multiindex of order 1.

(ii) Given a multiindex α , we define

$$\begin{aligned} D^\alpha u(x) &= \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} \\ &= \frac{\partial^{\alpha_1 + \alpha_2 + \dots + \alpha_n} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} \\ &= \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdot \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdot \dots \cdot \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u(x) \\ &= \partial x_1^{\alpha_1} \cdot \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n} u(x) \end{aligned}$$

e.g. (a) if take $\alpha = (1, 0, \dots, 0)$

$$\text{then } D^\alpha u(x) = \frac{\partial}{\partial x_1} u(x) = \frac{\partial y}{\partial x_1}$$

(b) take $\alpha = (0, 1, 0, \dots, 0)$

where $D^\alpha u(x) = \frac{\partial u(x)}{\partial x_2}$

(c) $\alpha = (1, 1, 0, \dots, 0)$

then $D^\alpha u(x) = \frac{\partial^2 u}{\partial x_1 \partial x_2}$

(d) $\alpha = (0, 0, 1, 1, 1, 1, \dots, 0)$

then $D^\alpha u(x) = \frac{\partial^4 u}{\partial x_3 \partial x_4 \partial x_5 \partial x_6}$

(ii) if k is a non-negative integer, then

$$D^k u(x) = \{ D^\alpha u(x) \mid |\alpha| = k \}$$

i.e. The set of all partial derivatives of order k . Assigning some order to the various partial derivatives, we can also regard $D^k u(x)$ as a point in $\mathbb{R}^{n^k}$

e.g. (a) Let $k=1$

then $D^1 u(x) = \{ D^\alpha u(x) \mid |\alpha| = 1 \}$

$$= \{ D^\alpha u(x) \mid \alpha_1 + \alpha_2 + \dots + \alpha_n = 1 \}$$

i.e. $\alpha \in \{ (1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 1) \}$

$$= \left\{ \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \dots, \frac{\partial u}{\partial x_n} \right\}$$

= set of all first order partial derivative u .

b) Let $k=2$

⑦

$$\text{then } D^2 u(x) = \{ D^\alpha u(x) \mid |\alpha|=2 \}$$

$$= \{ D^\alpha u(x) \mid \alpha_1 + \alpha_2 + \dots + \alpha_n = 2 \}$$

= The set of all second order partial derivatives of u

$$= \left(\frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \dots, \frac{\partial^2 u}{\partial x_2 \partial x_1}, \frac{\partial^2 u}{\partial x_2^2}, \dots, \frac{\partial^2 u}{\partial x_2 \partial x_n}, \dots, \frac{\partial^2 u}{\partial x_n \partial x_1}, \frac{\partial^2 u}{\partial x_n \partial x_2}, \dots, \frac{\partial^2 u}{\partial x_n^2} \right)$$

$\alpha \in \{ (2, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (0, 1, 1, 0, \dots, 0), \dots, (0, 0, \dots, 2) \}$

$$(iv) |D^k u| = \left(\sum_{|\alpha|=k} |D^\alpha u|^2 \right)^{1/2}$$

(v) special case :- if $k=1$

We regard the elements of $Du = Du = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$ as arranged in a vector and Du is called gradient vector.

* if $k=2$

then we regard the elements of $D^2 u$ as been arranged in matrix.

$$D^2 u = \begin{bmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \frac{\partial^2 u}{\partial x_2 \partial x_1} & \frac{\partial^2 u}{\partial x_2^2} & \dots & \frac{\partial^2 u}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \frac{\partial^2 u}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 u}{\partial x_n^2} \end{bmatrix}_{n \times n}$$

is called Hessian matrix.

(vi) Trace of D^2u :-

$$\Delta u = \text{trace of } D^2u$$

$$= \text{tr}(D^2u)$$

$$= \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}$$

$$= \sum_{j=1}^n u_{x_j x_j} = \text{Laplacian of } u$$

* We sometimes employ a subscript attached to D, D^2 etc to denote the variables being differentiated

e.g. if $u = u(x, y)$, $x, y \in \mathbb{R}^n$

then $D_x u$ denotes partial derivative of u w.r.t x .

(F) Function space:- for $U \subseteq \mathbb{R}^n$

(i) $C(U) = \{u \mid u: U \rightarrow \mathbb{R} \text{ is continuous}\}$

(ii) $C(\bar{U}) = \{u \in C(U) \mid u \text{ is uniformly continuous on bounded subset of } U\}$

(iii) $C^k(U) = \{u: U \rightarrow \mathbb{R} \mid u \text{ is } k\text{-times continuously differentiable}\}$

(iv) $C^k(\bar{U}) = \{u \in C^k(U) \mid D^\alpha u \text{ is uniformly continuous on bounded subset of } U \text{ for all } |\alpha| \leq k\}$.

if $u \in C^k(\bar{U})$, then $D^\alpha u$ continuously extends to $\bar{U}$ for each $|\alpha| \leq k$.

(V) $C^\infty(U) = \{u: U \rightarrow \mathbb{R} \mid u \text{ is infinitely differentiable}\}$
 $= \text{set of all smooth functions defined on } U$
 $= \bigcap C^k(U)$

and $C^\infty(\bar{U}) = \bigcap_{k=0}^{\infty} C^k(\bar{U})$

* Partial differential eqn:-

A partial diff eqn is an eqn involving an unknown function of two or more variables and certain of its derivative.

kth-order P.D.E:-

fix an integer $k \geq 1$ and let U denote open subset of $\mathbb{R}^n$. An expression of the form

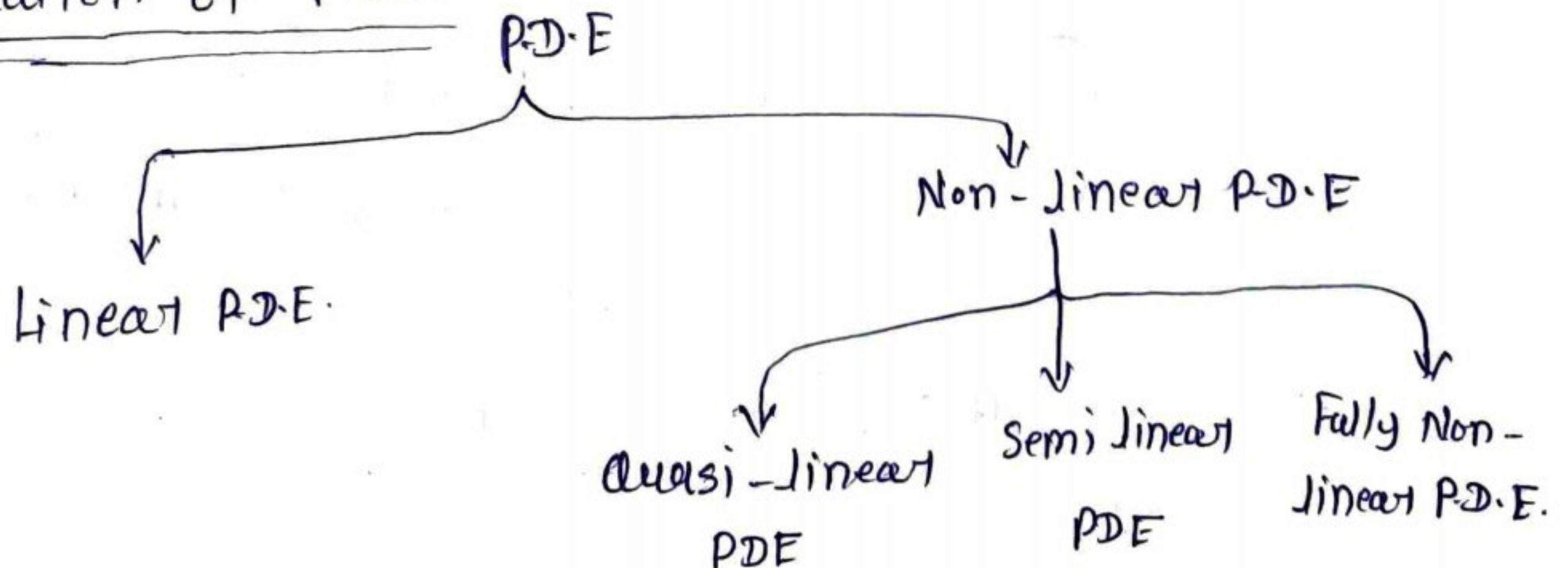
$$F(D^k u(x), D^{k-1} u(x), \dots, D u(x), u(x), x) = 0$$

where $x \in U$, is called k^{th} -order P.D.E. ———— ①

where $F: \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times U \rightarrow \mathbb{R}$ is given

and $u: U \rightarrow \mathbb{R}$ is unknown.

Classification of P.D.E:-



1) Linear PDE:- The partial differential equation

$$F(D^k u, D^{k-1} u, \dots, Du(x), u(x), x) = 0$$

where $x \in U$, $U \subseteq \mathbb{R}^n$, $k \geq 1$

is called linear if it has the form

$$\sum_{|\alpha| \leq k} q_\alpha(x) D^\alpha u = f(x)$$

for given functions $q_\alpha(x)$ and f .

i.e., if the dependent variable and ~~its~~ all its partial derivatives occur linearly with coefficients depending on the independent variable ~~and~~ alone in any PDE such PDE is called linear PDE otherwise non-linear.

e.g. ① $a(x, y) u_x + b(x, y) u_y + c(x, y) u = f(x, y)$

② $2u_{xx} + 4u_{xt} + 3u_{tt} + 4u_x + \cos t = 0$

③ $2u_{xx} - 3u_t + 4u_x = 0$

2) Quasi-linear PDE:- The partial differential equation

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0$$

is called quasi-linear if it has of the form

$$\sum_{|\alpha| = k} q_\alpha(D^{k-1} u + \dots, Du, u, x) D^\alpha u + q_0(D^{k-1} u, \dots, Du, u, x) = 0$$

i.e. A PDE is called quasi-linear if all the terms with highest order derivatives of dependent variable occur linearly, that is coeff of such terms are function of only lower order derivatives of dependent

variable.

(lower order derivative can occur in any manner)

e.g. (i) $a(x, y, u) u_x + b(x, y, u) u_y = f(x, y, u)$

for $z = z(x, y)$, $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$

$\therefore$ Q.L. P.D.E. is $p(x, y, z) p + Q(x, y, z) q = R(x, y, z)$

(ii) $(x^2 + z^2)p - xyq = z^3x + y^2$

(iii) $xzp + z^2yq = (x+y)z^2 + x^2 + y^2$

(iv) $x^2z^3p + xyzq = x^2yz^4 + x^3y^3$

(v) $xyu u_x + x^2u^2u_y = u^2xy$

3) Semi-Linear P.D.E :- The partial differential equation

$$F(D_{xx}^k, D_{xy}^{k-1}, \dots, D_{yy}^k, u_x, u_y, x) = 0$$

is called semilinear if it has of the form

$$\sum_{|\alpha|=k} a_{\alpha}(x) D^{\alpha} u + a_0(D_{xx}^{k-1}, D_{xy}^{k-1}, \dots, D_{yy}^k, u_x, u_y, x) = 0$$

In particular, this means that quasilinear PDE's are called semilinear P.D.E in which the coefficient of the terms involving the highest order derivatives of dependent variables depends only on the independent variable alone.

e.g. (i) $a(x, y) u_x + b(x, y) u_y = f(x, y, u)$

for $z = z(x, y)$, $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$

The S.L. P.D.E. is $P(x, y) p + Q(x, y) q = R(x, y, z)$

(i)

$$yp - xq = xz^2 + x$$

(ii)

$$xp + yq - (x+y)z^2 = x^2y^2 \Leftrightarrow xp + yq = (x+y)z^2 + x^2y^2$$

(iv)

$$p + (x+y)q = z^3xy + ex$$

(v)

4) Fully Non-Linear P.D.E:- The partial differential eqⁿ

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0$$

is fully non-linear if the highest order derivatives of dependent variable (i.e. of u) appear non-linearly in the eqⁿ (i.e. of first order)

e.g. $u_x^2 + u_y^2 = 1$ is fully non-linear.

* Throughout the course, $x \in U$, $U \subseteq \mathbb{R}^n$ is open, $t \geq 0$

Also $Du = D_x u = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$ denotes the gradient of u w.r.t x and t denotes the time

Some more example of Linear P.D.E:-

1) Laplace Equation:-

$$\Delta u = 0$$

$$\text{i.e. } \sum_{i=1}^n u_{x_i} x_i = 0$$

$$\text{i.e. } \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

Which is second order P.D.E and is linear.

2) Linear Transport Equation:-



$$u_t + \sum_{j=1}^n b_j u_{x_j} = 0$$

$$\text{i.e. } u_t + b_1 u_{x_1} + b_2 u_{x_2} + \dots + b_n u_{x_n} = 0$$

$$\text{i.e. } u_t + (b_1, b_2, \dots, b_n) \cdot (u_{x_1}, u_{x_2}, \dots, u_{x_n}) = 0$$

$$u_t + b \cdot \nabla u = 0$$

where $b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$

be a fixed vector.

Transport Eqⁿ:-

The most simple PDE is transport eqⁿ,
with ^{constant} coefficients.

The transport eqⁿ is

$$u_t + b \cdot \nabla u = 0 \quad \text{--- (1) in } \mathbb{R}^n \times (0, \infty)$$

where $b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ be a fixed vector

and $u: \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ is the unknown function

$$u = u(x, t), \quad \text{where } x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

denotes a typical point in space and t denotes typical time.

Here $\nabla u = \nabla_x u = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$ is written for gradient of u w.r.t x .

Solⁿ:-

Let u be the solⁿ of the P.D.E.

$$u_t + b \cdot \nabla u = 0 \quad \text{--- (2)}$$

$$\Rightarrow u_t + (b_1, b_2, \dots, b_n) \cdot (u_{x_1}, u_{x_2}, \dots, u_{x_n}) = 0$$

$$\Rightarrow 1 \cdot u_t + b_1 u_{x_1} + b_2 u_{x_2} + \dots + b_n u_{x_n} = 0$$

$$\Rightarrow (b_1, b_2, \dots, b_n, 1) \cdot (u_{x_1}, u_{x_2}, \dots, u_{x_n}, u_t) = 0$$

$$\Rightarrow (b \cdot 1) \cdot \nabla' u = 0 \quad \text{--- (3) where } b = (b_1, b_2, \dots, b_n)$$

$$\& \nabla' u = (u_{x_1}, u_{x_2}, \dots, u_{x_n}, u_t)$$

eqn (8) implies the directional derivatives of u in the direction $(b, 1)$ vanish. (14)

We exploit this insight by fixing any point

$$(x, t) \in \mathbb{R}^n \times (0, \infty) \text{ and}$$

Let us define, $z: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} z(s) &= u(x + sb, t + s) \quad , \text{ where } s \in \mathbb{R} \\ &= u(x_1 + sb_1, x_2 + sb_2, \dots, x_n + sb_n, t + s) \end{aligned}$$

$$\text{say } z(s) = u(x'_1, x'_2, \dots, x'_n, t') \quad \text{where } x'_i = x_i + sb_i$$

$$t' = t + s$$

differentiating w.r.t 's' both sides, we have

$$\begin{aligned} \dot{z}(s) &= \frac{dz}{ds} = \frac{d}{ds} (u(x'_1, x'_2, \dots, x'_n, t')) \\ &= \frac{\partial u}{\partial x'_1} \frac{dx'_1}{ds} + \frac{\partial u}{\partial x'_2} \frac{dx'_2}{ds} + \dots + \frac{\partial u}{\partial x'_n} \frac{dx'_n}{ds} + \frac{\partial u}{\partial t'} \frac{dt'}{ds} \\ &= \frac{\partial u}{\partial x'_1} b_1 + \frac{\partial u}{\partial x'_2} b_2 + \dots + \frac{\partial u}{\partial x'_n} b_n + \frac{\partial u}{\partial t'} \cdot 1 \\ &= (b_1, b_2, \dots, b_n) \left(\frac{\partial u}{\partial x'_1} \dots \frac{\partial u}{\partial x'_n} \right) + \frac{\partial u}{\partial t'} \\ &= b \cdot Du(x + sb, t + s) + u_t(x + sb, t + s) \\ &= 0 \quad \left(\begin{array}{l} \text{using (2)} \\ \because u \text{ is soln of given P.D.E (2)} \end{array} \right) \end{aligned}$$

$$\text{i.e. } \dot{z}(s) = 0$$

$$z(s) = \text{constant}$$

$\Rightarrow$ u is constant on the line passing through (x, t) with direction $(b, 1)$. Thus for each point (x, t) , u is constant on the line passing through (x, t) with direction $(b, 1)$.

Hence if we know the value of u at any point on each such line then we know its value everywhere in $\mathbb{R}^n \times (0, \infty)$

Initial value problem :-

Find solⁿ of the homogeneous linear initial value problem

$$\left. \begin{aligned} u_t + b \cdot Du &= 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g & \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned} \right\} \text{--- (1)}$$

where $b \in \mathbb{R}^n$ and $g: \mathbb{R}^n \rightarrow \mathbb{R}$ is ~~is known~~ ~~known~~ known
compute u .

Solⁿ:- Let (x, t) be any point in $\mathbb{R}^n \times (0, \infty)$ and let u be solⁿ of given P.D.E.

Let us define $z: \mathbb{R} \rightarrow \mathbb{R}$ by

$$z(s) = u(x + sb, t + s), \quad s \in \mathbb{R}$$

$$\dot{z}(s) = b \cdot Du(x + sb, t + s) + u_t(x + sb, t + s)$$

as we done in prev. proof)

$$= 0 \quad [u \text{ is solⁿ of (1)}]$$

$$\dot{z}(s) = 0$$

$\Rightarrow z$ is a constant fn of s

$\Rightarrow u$ is constant on the line passing through (x, t) with direction $(b, 1)$, which (line) is represented parametrically by

$$(x, t) + s(b, 1)$$

$$\text{i.e. } (x + sb, t + s), \quad s \in \mathbb{R}$$

This line hits the plane $\Gamma: \mathbb{R}^n \times \{t=0\}$ when $s = -t$,
at the point $(x - tb, 0)$

since u is constant along the line, and

$$u(x - tb, 0) = g(x - tb)$$

$$[\because u(x, t) = g(x) \text{ on } \mathbb{R}^n \times \{t=0\}]$$

$\therefore$ we deduce $u(x, t) = g(x - tb)$ --- (2)

where $x \in \mathbb{R}^n, t \geq 0$

∴ if eqn ① has a soln u , then it must be given by eqn ②. (16)

conversely, it is easy to check directly if g is C^1 , then u defined by ② is a soln of eqn ①

$$\text{from eqn ② } u(x, t) = g(x - tb)$$

$$\Rightarrow u_t = -b Dg(x - tb)$$

$$\text{and } Du = Dg(x - tb)$$

$$\therefore \text{ using in eqn ① } u_t + b Du = -b Dg + b(Dg) = 0$$

$$\Rightarrow u = g(x - tb) \text{ is a soln of ①}$$

and for $t=0$

$$u(x, 0) = g(x) \text{ in } \mathbb{R}^n.$$

** Non-homogeneous Transport Problem :-

Thm:- Solve non-homogeneous transport problem

$$\left. \begin{aligned} u_t + b \cdot Du &= f \text{ in } \mathbb{R}^n \times (0, \infty) \\ u &= g \text{ on } \mathbb{R}^n \times \{t=0\} \end{aligned} \right\} \text{--- ①}$$

where $b \in \mathbb{R}^n$ is fixed and $f: \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ and $g: \mathbb{R}^n \rightarrow \mathbb{R}$ are known functions

Soln:- Let (x, t) be any point in $\mathbb{R}^n \times (0, \infty)$ and u be soln of given P.D.E ①

Let us define a function $z: \mathbb{R} \rightarrow \mathbb{R}$ by

$$z(s) = u(x + sb, t + s), \quad s \in \mathbb{R} \text{ --- ②}$$

differentiating w.r.t s both sides

$$\dot{z}(s) = b \cdot Du(x+sb, t+s) + u(x+sb, t+s)$$

$$= f(x+bs, t+s)$$

[$\because u$ is soln of PDE (2)]

Integrating both sides w.r.t s and take limit $-t$ to 0 we get

$$z(s) \Big|_{-t}^0 = \int_{-t}^0 f(x+bs, t+s) ds$$

$$\Rightarrow z(0) - z(-t) = \int_{-t}^0 f(x+bs, t+s) ds$$

$$\Rightarrow u(x, t) - u(x-tb, 0) = \int_{-t}^0 f(x+bs, t+s) ds \quad [\text{using eqn (2)}]$$

$$\begin{array}{l|l} \text{put } t+s = s' & s = -t \Rightarrow s' = 0 \\ s = s' - t & s = 0 \Rightarrow s' = t \\ ds = ds' & \end{array}$$

$$\Rightarrow u(x, t) - g(x-tb) = \int_0^t f(x+(s'-t)b, s') ds' \quad [\because u=g \text{ on } \mathbb{R}^n \times \{t=0\}]$$

$$\Rightarrow u(x, t) = g(x-tb) + \int_0^t f(x+(s'-t)b, s') ds'$$

$$\Rightarrow u(x, t) = g(x-tb) + \int_0^t f(x+(s-t)b, s) ds \quad \text{--- (3)}$$

(renaming the variable from s' to s)

where $x \in \mathbb{R}^n$ and $t \geq 0$

Hence $u(x, t)$ given by eqn (3) is the required soln.

** Laplace Eqn :-

The Laplace eqn is given by

$$\Delta u = 0$$

$$\text{or } \sum_{j=1}^n u_{x_j} x_j = 0$$

Here $x \in U$ and unknown function $u: \bar{U} \rightarrow \mathbb{R}$

where $U \subseteq \mathbb{R}^n$ is a open set

Harmonic Function

A C^2 function u satisfying the Laplace eqⁿ, $\Delta u = 0$ is called Harmonic function.

Theorem Derive the fundamental solution of Laplace eqⁿ.

Proof:- Consider the Laplace eqⁿ $\Delta u = 0$ in $\mathbb{R}^n$
 $\text{or } \sum_{i=1}^n u_{x_i x_i} = 0$
 where $U \subseteq \mathbb{R}^n$ which is open set and $u: \bar{U} \rightarrow \mathbb{R}$ is unknown.

it is advisable to search for radial solⁿ
 i.e function of $r = |x|$

$\therefore$ We may consider that $u(x)$, the solⁿ of Laplace eqⁿ (1) is

$$u(x) = v(r) \quad \text{--- (2)}$$

$$\text{where } r = |x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

and v is to be selected (if possible)

so that $\Delta u = 0$ holds.

$$\text{since } r = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

$$\text{for } 1 \leq i \leq n, \quad \frac{\partial r}{\partial x_i} = \frac{1}{2} (x_1^2 + x_2^2 + \dots + x_n^2)^{-1/2} \cdot 2x_i$$

$$= \frac{x_i}{(x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}} = \frac{x_i}{r} \quad \text{--- (3)}$$

$$\text{Now } u(x) = v(r)$$

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x_i}$$

(using chain rule)

$$\Rightarrow \frac{\partial u}{\partial x_i} = v'(r) \cdot \frac{x_i}{r}$$

(using (3))

$$\Rightarrow \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left(v'(r) \cdot \frac{x_i}{r} \right)$$

$$= v''(x) \cdot \frac{\partial x}{\partial x_i} \cdot \frac{\partial x_i}{\partial x} + v'(x) \frac{\partial}{\partial x_i} \left(\frac{x_i}{x} \right)$$

$$= v''(x) \cdot \frac{x_i}{x^2} \cdot \frac{x_i}{x} + v'(x) \left(\frac{1}{x} - \frac{x_i}{x^2} \left(\frac{\partial x}{\partial x_i} \right) \right)$$

(using ⑥)

$$= v''(x) \left(\frac{x_i}{x} \right)^2 + v'(x) \left(\frac{1}{x} - \frac{x_i}{x^2} \cdot \frac{x_i}{x} \right)$$

$$= v''(x) \left(\frac{x_i}{x} \right)^2 + v'(x) \left(\frac{1}{x} - \frac{x_i^2}{x^3} \right)$$

$$u_{x_i x_i} = v''(x) \left(\frac{x_i}{x} \right)^2 - v'(x) \frac{x_i^2}{x^3} + \frac{v'(x)}{x}$$

taking summation from 1 to n both sides

$$\sum_{i=1}^n u_{x_i x_i} = \frac{v''(x)}{x^2} \sum_{i=1}^n x_i^2 - \frac{v'(x)}{x^3} \sum_{i=1}^n x_i^2 + \frac{v'(x)}{x} \sum_{i=1}^n 1$$

$$= \frac{v''(x)}{x^2} \cdot x^2 - \frac{v'(x)}{x^3} \cdot x^2 + \frac{v'(x)}{x} \cdot n$$

$$\Delta u = v''(x) - \frac{v'(x)}{x} + \frac{v'(x)}{x} n$$

$$= v''(x) + (n-1) \frac{v'(x)}{x}$$

Hence $\Delta u = 0$

iff $v''(x) + (n-1) \frac{v'(x)}{x} = 0 \Rightarrow v''(x) = (1-n) \frac{v'(x)}{x}$

if $v'(x) \neq 0$, then $\frac{v''(x)}{v'(x)} = \frac{1-n}{x}$

Integrating w.r.t 'x', we get

$$\log(v'(x)) = (1-n) \log x + \log a$$

[where a = constant]

$$= \log a x^{(1-n)}$$

$\Rightarrow$

$$v'(x) = a x^{(1-n)} = \frac{a}{x^{n-1}}$$

~~Again integrating~~

$\Rightarrow$

$$dv = \frac{a}{x^{n-1}} dx$$

$$v = \begin{cases} a \log x + c & \text{if } n=2 \\ \frac{a}{(2-n)x^{n-2}} + c & \text{if } n \geq 3 \end{cases}$$

where a & c are constants. (3)

consequently, if $n > 0$, we have

$$v(r) = \begin{cases} a \log r + c & \text{if } n=2 \\ \frac{b}{r^{n-2}} + c & \text{if } n \geq 3 \end{cases} \quad \text{where } b = \frac{a}{2-n}$$

Defⁿ:

The function

$$\Phi(x) = \begin{cases} -\frac{1}{2n} \log |x| & \text{if } n=2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} & \text{if } n \geq 3 \end{cases}$$

defined for $x \in \mathbb{R}^n$, $x \neq 0$

is the fundamental solⁿ of Laplace's eqⁿ.

Mean Value Formula for Laplace Equation:-

statement:- if $u \in C^2(U)$ is harmonic, then

$$u(x) = \int_{\partial B(x,r)} u ds = \int_{B(x,r)} u dy$$

for each ball $B(x,r) \subseteq U$

This formula declares that $u(x)$ equals both the averages of u over the sphere $\partial B(x,r)$ and the average of u over the entire ball $B(x,r)$ provided $B(x,r) \subseteq U$

Proof:

It is given that $u \in C^2(U)$ and u is harmonic.

Now define a function, $\Delta u = 0$

$$\phi(r) = \int_{\partial B(x,r)} u(y) dS(y) \quad \text{--- (1)}$$

Now if we write $y = x + z$, where z is the unit vector, i.e. $|z| = 1$

Then we have, $\phi(r) = \int_{\partial B(0,1)} u(x+z) dS(z)$

(20)

$$\left[\begin{aligned} \because y = x + z \Rightarrow y - x = z \Rightarrow |z| &= \left| \frac{y-x}{r} \right| \\ \Rightarrow |z| &= \left| \frac{r}{r} \right| = 1 \\ \text{centre} &= 0, \text{ radius} = 1 \end{aligned} \right. \begin{aligned} &\text{Here } y \in \partial B \\ &x \rightarrow \text{centre} \end{aligned}$$

differentiating both sides w.r.t 'r', we have

$$\phi'(r) = \int_{\partial B(0,r)} Du(x+z) \cdot z \, dS(z)$$

$$= \int_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} \cdot dS(y)$$

$$= \int_{\partial B(x,r)} Du(y) \cdot v \, dS(y) \quad \left[\begin{aligned} &\text{where } v = \frac{y-x}{r} \\ |v| &= \frac{|y-x|}{r} = \frac{r}{r} = 1 \end{aligned} \right]$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} Du(y) \cdot v \, dS(y)$$

$$\left[\begin{aligned} &\boxed{v = \frac{y-x}{r}} \\ &\because y \in \partial B(x,r) \\ &\Rightarrow |y-x| = r \end{aligned} \right]$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial v} \, dS(y) \quad \left[\begin{aligned} &\because Du \cdot v = \frac{\partial u}{\partial v} \\ &\text{direction derivative of } u \\ &\text{in the direction of } v \end{aligned} \right]$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u \, dy$$

$$\left[\begin{aligned} \because \int_0^r \Delta u \, dy &= \int_0^r \frac{\partial u}{\partial v} \, dv, \text{ where } v \text{ is the unit vector} \\ &\text{(Green's formulas)} \end{aligned} \right]$$

$$= \frac{r}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u \, dy$$

$$= \frac{r}{n} \int_{B(x,r)} \Delta u \, dy \quad \left[\begin{aligned} &\because \int_{B(x,r)} f \, dy = \frac{1}{\alpha(n)r^n} \int_{\partial B(x,r)} f \, dS \end{aligned} \right]$$

$$= 0$$

$$[\because \Delta u = 0]$$

$$\Rightarrow \phi(x) = 0$$

Hence ϕ is a constant function. i.e ϕ is independent of x

[$\therefore$ we can take ϕ be some function of some variable say t and take $t \rightarrow 0$
 such that ϕ becomes ind. of variable t -e

$$\Rightarrow \phi(x) = \lim_{t \rightarrow 0} \int_{\partial B(x,t)} u(y) dS(y) = u(x) \quad \text{--- (2)}$$

By (1) & (2), we have

$$u(x) = \int_{\partial B(x,t)} u(y) dS(y) \quad \text{--- (3)}$$

$$\text{Now } \int_{B(x,t)} u dy = \int_0^t \left(\int_{\partial B(x,s)} u dS(y) \right) ds$$

$$= \int_0^t \left(n \alpha(n) s^{n-1} \int_{\partial B(x,s)} u ds \right) ds$$

$$= \int_0^t n \alpha(n) s^{n-1} u(x) ds$$

$$= n \alpha(n) u(x) \int_0^t s^{n-1} ds$$

$$= n \alpha(n) u(x) \left[\frac{s^n}{n} \right]_0^t = n \cdot \alpha(n) \cdot u(x) \cdot \frac{t^n}{n}$$

$$\Rightarrow \int_{B(x,t)} u dy = \alpha(n) \cdot u(x) \cdot t^n$$

$$\Rightarrow u(x) = \frac{1}{\alpha(n) t^n} \int_{B(x,t)} u dy$$

$$\Rightarrow u(x) = \int_{B(x,t)} u dy \quad \text{--- (4)}$$

from (3) & (4)

$$u(x) = \int_{\partial B(x,t)} u ds = \int_{B(x,t)} u dy$$

$\therefore$ In the surphuse of sphere $\partial B(x,t)$ when $t \rightarrow 0$
 $\frac{t}{x} \rightarrow 0 \quad 0 \quad 0 \quad \dots$
 $\therefore$ average over surface $\partial B(x,t)$ when $t \rightarrow 0$ is nothing but value of u at x i.e $u(x)$

(using B)

Hence proved

converse to Mean value property for Laplace Eqn :-

Statement:- if $u \in C^2(U)$ satisfies

$$u(x) = \int_{\partial B(x,r)} u \, ds$$

for each ball $B(x,r) \subseteq U$, then u is harmonic

Proof:- Suppose u is not harmonic in U

$\Rightarrow \exists$ some ball $B(x,r) \subseteq U$ s.t. $\Delta u > 0$ or $\Delta u < 0$ over the ball $B(x,r)$ } — (*)

consider $\phi(r) = \int_{\partial B(x,r)} u(y) \, dS(y)$ — (1)

since $\int_{\partial B(x,r)} u(y) \, dS(y) = u(x)$ — (1') [given]

$\therefore \int_{\partial B(x,r)} u(y) \, dS(y)$ is independent of r .

$$\phi(r) = u(x)$$

(using (1) & (1'))

$\Rightarrow \phi(r)$ is independent of r

$\Rightarrow \phi'(r) = 0$ — (2)

But $\phi'(r) = \frac{1}{r} \int_{\partial B(x,r)} \Delta u \, dy$ — (3)

$\because \phi(r) = \int_{\partial B(x,r)} u(y) \, dS(y)$ — (4)

$\because y \in \partial B(x,r) \Rightarrow |y-x|=r$

if we write $y = x + rz$ where z is unit vector i.e. $|z|=1$

$\therefore \frac{y-x}{r} = z \Rightarrow |z| = \left| \frac{y-x}{r} \right| = \frac{|x|}{r} \geq 1$, centre $\rightarrow x$ radius $= r$

∴ We have

$$\phi(x) = \int_{\partial B(x,1)} u(x+z) dS(z)$$

diff both sides w.r.t 'x', we have

$$\phi'(x) = \int_{\partial B(x,1)} Du(x+z) \cdot z dS(z)$$

$$= \int_{\partial B(x,1)} Du(y) \cdot \frac{y-x}{1} dS(y)$$

$$= \int_{\partial B(x,1)} Du(y) \cdot v dS(y) \quad \left[\begin{array}{l} \text{where } v = \frac{y-x}{1} \\ |v| = 1 \text{ i.e. } v \text{ is unit} \\ \text{normal vector} \end{array} \right]$$

$$= \frac{1}{n\alpha(n)1^{n-1}} \int_{\partial B(x,1)} Du(y) \cdot v dS(y)$$

$$= \frac{1}{n\alpha(n)1^{n-1}} \int_{\partial B(x,1)} \frac{\partial u}{\partial v} dS(y) \quad \left[Du \cdot v = \frac{\partial u}{\partial v} \right]$$

$$= \frac{1}{n\alpha(n)1^{n-1}} \int_{\partial B(x,1)} \Delta u dy$$

$$= \frac{1}{n\alpha(n)1^n} \int_{B(x,1)} \Delta u dy$$

$$\phi'(x) = \frac{1}{n} \int_{B(x,1)} \Delta u dy$$

$$\left[\begin{array}{l} \therefore \int_U \Delta u dy = \int_U \frac{\partial u}{\partial v} dS \\ \text{where } v \text{ is unit vector} \\ \text{Green formula} \end{array} \right]$$

$$\text{from ② \& ③} \quad \frac{1}{n} \int_{B(x,1)} \Delta u dy = 0 \quad [\because \phi'(x) = 0]$$

$$\Rightarrow \int_{B(x,1)} \Delta u dy = 0 \quad \text{--- (**)}$$

This holds for each ball $B(x,1) \subseteq U$ but if we consider the ball as given in (*), we get

$$\int_{B(x_1, r)} \Delta u \, dy > 0 \text{ or } \int_{B(x_1, r)} \Delta u \, dy < 0$$

(18)

which is a contradiction to (17) - Hence our supposition is wrong

$$\therefore \Delta u = 0 \text{ for every } B(x_1, r) \subseteq U$$

$$\Rightarrow \Delta u = 0 \text{ in } U$$

i.e. u is harmonic

Properties of Harmonic function:-

Assume for the following that $U \subseteq \mathbb{R}^n$ is open and

bounded. ~~strong~~ Maximum

strong Maximum Principle:-

We begin with assumption that a

harmonic function must attain its maximum on the boundary and cannot attain maximum in the interior of a connected region unless it is constant.

Thm:- (strong Maximum Principle):-

Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic within

U . Then

$$(i) \quad \max_{\bar{U}} u = \max_{\partial U} u$$

(ii) Furthermore, if U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u,$$

then, u is constant within U .

Proof:- first we ~~have to~~ prove (i)

suppose U is connected

$\Rightarrow$ if $\exists$ any subset of U which is both open and closed ⁽²⁶⁾, then it is either empty or U .

Also, it is given that $\exists$ a point $x_0 \in U$ such that

$$u(x_0) = \max_U u$$

Let $M = u(x_0)$ then $M = \max_U u$

Let $U_M = \{x \in U \mid u(x) = M\}$ ——— ①

since $u(x_0) = M$ and $x_0 \in U$

$\Rightarrow x_0 \in U_M$

$\Rightarrow U_M$ is non-empty (using ①)

$$\begin{aligned} \text{Now } U_M &= \{x \in U \mid u(x) = M\} \\ &= \{x \in U \mid u(x) \in \{M\}\} \\ &= u^{-1}(\{M\}) \quad \text{————— ②} \end{aligned}$$

since $\{M\}$ is a finite subset of $\mathbb{R}$ and every finite subset of $\mathbb{R}$ is closed,

$\therefore \{M\}$ is a closed subset of $\mathbb{R}$

$u: U \rightarrow \mathbb{R}$ is continuous

$\Rightarrow u^{-1}(\{M\})$ is a closed subset of U

$\Rightarrow U_M$ is a closed subset of U

We claim that U_M is an open subset of U .

Let $x \in U_M$ then $u(x) = M$

$$\text{i.e. } u(x) = M = \max_U u$$

Let $0 < r < \text{dist}(x, \partial U)$

then $B(x, r) \subseteq U$

[$\therefore$ distance of x from ∂U
Here $r \in$ distance between x & ∂U exist as U is open
 $\Rightarrow U \cap \partial U = \emptyset$]

since $u \in C^2(U)$ is harmonic

$\therefore$ By Mean value formula

$$u(x) = \int_{B(x,r)} u dy$$

(27)

$$i.e. M = \int_{B(x,r)} u dy$$

$$\text{since } M = \max_U u$$

$$\therefore u(y) \leq M \quad \forall y \in \bar{U}$$

$$\text{since } B(x,r) \subseteq U \subseteq \bar{U}$$

$$\therefore u(y) \leq M \quad \forall y \in B(x,r)$$

$$\Rightarrow \int_{B(x,r)} u dy \leq M$$

The equality will hold iff $u=M$ within $B(x,r)$

$$i.e. u(y) = M \quad \forall y \in B(x,r)$$

$$\Rightarrow B(x,r) \subseteq U_M \quad \text{--- (3) using (1)}$$

closed ball

$$\text{Since } B^o(x,r) \subseteq B(x,r) \quad \text{--- (4)}$$

$$\text{from (3) \& (4) } B^o(x,r) \subseteq U_M$$

Thus for each $x \in U_M$, $\exists r > 0$ s.t

$$B^o(x,r) \subseteq U_M$$

$\Rightarrow U_M$ is an open subset of U

since U is connected, $U_M = U$

$$\text{Let } x \in U \Rightarrow x \in U_M \Rightarrow u(x) = M$$

As x was chosen arbitrary, thus

$$u(x) = M \quad \forall x \in U$$

$\Rightarrow u$ is constant within U .

Now we prove (i)

$$\max_U u = \max_{\partial U} u$$

Let $M = \max_{\bar{U}} u = \max \{ u(x) \mid x \in \bar{U} \}$ — (5)

then $M = u(x_0)$ for some $x_0 \in \bar{U}$ — (6)

$\because u: \bar{U} \rightarrow \mathbb{R}$ is continuous & $\bar{U}$ is closed.

since $\bar{U} = U \cup \partial U$ — (6)

using (6) as $x_0 \in \bar{U}$ i.e. $x_0 \in U \cup \partial U$

$\therefore$ either $M = u(x_0)$ for some $x_0 \in U$
or $M = u(x_0)$ for some $x_0 \in \partial U$

if $M = u(x_0)$ for some $x_0 \in \partial U$
i.e. $M = \max_{\partial U} u$

if $M = u(x_0)$ for some $x_0 \in U$

then by using above proof of (ii),
 u is constant with U .

i.e. $u(x) = M$ for $\forall x \in U$ — (7)

Let $a \in \partial U$

Since u is a continuous function.

$\therefore u(a) = \lim_{x \rightarrow a} u(x)$

Since $u(x) = M \forall x \in U$

$\therefore \lim_{x \rightarrow a} u(x) = M$

$\Rightarrow u(a) = M \forall a \in \partial U$ — (8)

As u gives constant value of every point of $\bar{U} = U \cup \partial U$
 $\Rightarrow$ this means value comes from either U or ∂U $\therefore \emptyset$
both the value of U & ∂U will be same

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$$y(u) = M \quad \forall u \in \overline{U}$$

i.e. in this case, maximum value attained by u is either in U or in ∂U is same

$$\text{i.e. } M = \frac{\text{Max } u}{\bar{v}} = \frac{\text{Max } u}{\partial v}$$

Remark:- In above principle:-

part(i) is the maximum principle for Laplace eqⁿ

& part (ii) is strong maximum principle.

Remark:- if we replace u by $-u$, we get minimum principle.

Theorem 1. Uniqueness of solⁿ to certain boundary value problem (B-V.P) for poisson eqⁿ

Statement:- Let $g \in C(\partial U)$, $f \in C(U)$. Then $\exists$ at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the B.V.P.

$$\left. \begin{array}{l} \Delta u = -f \quad \text{in } U \subseteq \mathbb{R}^n \\ u = g \quad \text{on } \partial U \end{array} \right\} \quad \text{--- } \textcircled{1}$$

Proof:- Let $\tilde{u}$ & $\tilde{v}$ be two solution of B.V.P ①

then $\Delta \tilde{u} = -f$ in V
 $\Delta \tilde{v} = -f$ in V } — 

and $\begin{matrix} \bar{u} = g \\ \bar{v} = g \end{matrix} \quad \left. \begin{matrix} \text{on } \partial U \\ \text{on } \partial U \end{matrix} \right\} \text{---} \textcircled{3}$

Let $w = \pm (\tilde{\alpha} - \tilde{\beta})$ be such that either w is

the or zero [i.e. $w(y) \geq 0 \quad \forall y \in \bar{U}$]

then in U

$$\begin{aligned}\Delta w &= \pm (\Delta \tilde{u} - \Delta \tilde{v}) \\ &= \pm (-f + f) = 0 \quad \text{using (2)}\end{aligned}$$

Also on ∂U , $w = \pm (\tilde{u} - \tilde{v})$

$$= \pm (g - g) = 0 \quad \text{using (3)}$$

Thus w satisfies $\Delta w = 0$ in U

& $w = 0$ on ∂U — (4)

$\Rightarrow w$ satisfies the Laplace eqn

i.e. w is harmonic

since $\tilde{u}, \tilde{v} \in C^2(U) \cap C(\bar{U})$

$$\Rightarrow w \in C^2(U) \cap C(\bar{U})$$

Hence by strong maximum principle.

$$\begin{aligned}\max_{\bar{U}} w &= \max_{\partial U} w \\ &= 0 \quad \text{using (4)}\end{aligned}$$

$$\Rightarrow \max_{\bar{U}} w = 0$$

$$\Rightarrow w = 0 \quad \text{on } \bar{U}$$

$$\Rightarrow \tilde{u} - \tilde{v} = 0$$

$$\Rightarrow \tilde{u} = \tilde{v}$$

$$\left[\begin{array}{l} \because w(y) \geq 0 \quad \forall y \in \bar{U} \\ \& \text{ since } \max w = 0 \text{ on } \bar{U} \\ \Rightarrow w = 0 \text{ on } \bar{U} \end{array} \right.$$

Thus $\exists$ atmost one solution of B.V.P (1)

Notations:-

$$L^\infty(U) = \{ u: U \rightarrow \mathbb{R} \mid u \text{ is Lebesgue measurable} \}$$

$$\text{and } \|u\|_{L^\infty(U)} < \infty$$

$$\text{where } \|u\|_{L^\infty(U)} = \text{ess. sup } |u|$$

$$= \text{glb } \{ M \mid |u(x)| \leq M \text{ a.e. on } U \}$$

$$= \text{glb } A$$

$$\text{where } A = \{ M \mid |u(x)| \leq M \text{ a.e. on } U \}$$

Remark:- ① $\|u\|_{L^\infty(U)} \leq M \quad \forall M \in A$

$$\text{i.e. } \|u\|_{L^\infty(U)} \leq M \quad \forall M \text{ s.t. } |u(x)| \leq M \text{ a.e. on } U$$

$$\textcircled{2} \quad |u(x)| \leq M \text{ a.e. } \forall M \in A$$

$$\Rightarrow |u(x)| \text{ is a lower bound of } A$$

$$\Rightarrow |u(x)| \leq \text{glb } M$$

$$M \in A$$

$$\left[\begin{array}{l} \because \text{L.b} \leq \text{g.l.b} \\ \text{i.e. } |u| \leq \|u\|_{L^\infty(U)} \end{array} \right]$$

$$\text{i.e. } |u(x)| \leq \|u\|_{L^\infty(U)}$$

Defⁿ:-

$$\text{Let } u \in C^1(\bar{U})$$

then we call $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$, the outward normal derivative of u , where $\nu = \{ \nu^1, \nu^2, \dots, \nu^n \}$ is outward unit normal vector.

~~Guess~~ Gauss Green's Theorem:-

Suppose U is a bounded open subset of $\mathbb{R}^n$ and $u \in C^1(\bar{U})$. Then

$$\int_U u_{x_j} dx = \int_{\partial U} u \cdot \nu_j ds$$

where $j=1, 2, \dots, n$

Theorem :-Local Estimate for Harmonic Functions:

(Estimates on derivatives)

Statement :- Assume u is harmonic in U

$$\text{Then } |D^\alpha u(x_0)| \leq \frac{C_k}{\pi^{n+k}} \|u\|_{L^2(B(x_0, r))} \quad \text{--- (*)}$$

for each ball $B(x_0, r) \subset U$ and for each multiindex α of order $|\alpha| = k$.

where $C_0 = \frac{1}{\alpha(n)}$, $C_k = \frac{(2^{n+k} n k)^k}{\alpha(n)}$, $k = 1, 2, 3, \dots$

Proof:- We will prove the result using induction on k .Case I:- when $k=0$

$$\Rightarrow |\alpha| = 0$$

$$\Rightarrow \alpha_1 + \alpha_2 + \dots + \alpha_n = 0$$

$$\Rightarrow \alpha_i = 0 \quad \forall \quad i = 1, 2, \dots, n$$

$$\Rightarrow D^\alpha u(x) = u(x)$$

$$\Rightarrow |D^\alpha u(x_0)| = |u(x_0)| \quad \text{--- (1)}$$

Since u is harmonic in U , therefore By mean value formula for Laplace eqn, we have

$$u(x_0) = \int_{B(x_0, r)} u(y) dy \quad \text{for each ball } B(x_0, r) \subset U$$

$$\Rightarrow |u(x_0)| = \left| \int_{B(x_0, r)} u(y) dy \right|$$

$$= \left| \frac{1}{\alpha(n) \cdot \pi^n} \int_{B(x_0, r)} u(y) dy \right|$$

$$\leq \frac{1}{\alpha(n) \cdot \pi^n} \int_{B(x_0, r)} |u(y)| dy$$

$$= \frac{C_0}{r^{n+1}} \|u\|_{L^1(B(x_0, r))}$$

$$\left[\because \|u\|_{L^p(U)} = \left(\int_U |u|^p dx \right)^{1/p} \right]$$

$$\Rightarrow |D^\alpha u(x_0)| \leq \frac{C_0}{r^{n+1}} \|u\|_{L^1(B(x_0, r))} \quad (\text{using (D)})$$

thus the result is true for $k=0$

Case II:- Let $k=1$

since u is harmonic in U ,

$$\therefore \Delta u = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

differentiating partially w.r.t x_i , we have

$$\frac{\partial^3 u}{\partial x_i \partial x_1^2} + \frac{\partial^3 u}{\partial x_i \partial x_2^2} + \dots + \frac{\partial^3 u}{\partial x_i \partial x_n^2} = 0$$

$$\frac{\partial^2}{\partial x_i^2} \left(\frac{\partial u}{\partial x_1} \right) + \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial u}{\partial x_2} \right) + \dots + \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial u}{\partial x_n} \right) = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x_i^2} (u_{x_1}) + \frac{\partial^2}{\partial x_i^2} (u_{x_2}) + \dots + \frac{\partial^2}{\partial x_i^2} (u_{x_n}) = 0$$

i.e. u_{x_i} satisfies the Laplace eqn $\forall i (1 \leq i \leq n)$

i.e. u_{x_i} is a harmonic function. $\forall i (1 \leq i \leq n)$

$$\text{Now } B(x_0, \frac{r}{2}) \subseteq B(x_0, r) \subseteq U$$

$\therefore$ By applying mean value formula for u_{x_i} , we have

$$u_{x_i}(x_0) = \frac{1}{|B(x_0, \frac{r}{2})|} \int_{B(x_0, \frac{r}{2})} u_{x_i}(y) dy$$

$$\begin{aligned} \Rightarrow |u_{x_i}(x_0)| &= \left| \frac{1}{|B(x_0, \frac{r}{2})|} \int_{B(x_0, \frac{r}{2})} u_{x_i}(y) dy \right| \\ &= \left| \frac{1}{2^n \pi \cdot (\frac{r}{2})^n} \int_{B(x_0, \frac{r}{2})} u_{x_i}(y) dy \right| \end{aligned}$$

(34)

$$= \left| \frac{2^n}{\alpha(n) \pi^n} \int_{\partial B(x_0, \frac{\pi}{2})} u \cdot v^j ds \right|$$

[By Gauss Green's Theorem]

$$\leq \frac{2^n}{\alpha(n) \pi^n} \int_{\partial B(x_0, \frac{\pi}{2})} |u| ds$$

[$\because v$ is a unit vector]

$$= \frac{2^n}{\alpha(n) \pi^2} \cdot n \alpha(n) \left(\frac{\pi}{2}\right)^{n-1} \int_{\partial B(x_0, \frac{\pi}{2})} |u| ds$$

$$= \frac{2^n}{\pi} \int_{\partial B(x_0, \frac{\pi}{2})} |u| ds$$

$$\leq \frac{2^n}{\pi} \|u\|_{L^\infty(\partial B(x_0, \frac{\pi}{2}))} \left[\begin{array}{l} \because |u(x)| \leq \|u\|_{L^\infty(U)} \quad \forall x \in U \\ \Rightarrow \int_U |u| ds \leq \|u\|_{L^\infty(U)} \end{array} \right]$$

$$\Rightarrow |u(x_0)| \leq \frac{2^n}{\pi} \|u\|_{L^\infty(\partial B(x_0, \frac{\pi}{2}))} \quad \text{--- (2)}$$

Now we claim that, $\|u\|_{L^\infty(\partial B(x_0, \frac{\pi}{2}))} \leq \frac{2^n}{\alpha(n) \pi^n} \|u\|_{L(B(x_0, \pi))}$

To prove this, let $x \in \partial B(x_0, \frac{\pi}{2})$

$$\text{then } B(x, \frac{\pi}{2}) \subset B(x_0, \pi) \subset U \quad \text{--- (3)}$$

$$\left[\begin{array}{l} \because \text{Let } y \in B(x, \frac{\pi}{2}) \Rightarrow d(y, x) \leq \frac{\pi}{2} \\ \text{consider } d(y, x_0) \leq d(y, x) + d(x, x_0) \\ \leq \frac{\pi}{2} + \frac{\pi}{2} = \pi \\ \Rightarrow y \in B(x_0, \pi) \\ \Rightarrow B(x, \frac{\pi}{2}) \subset B(x_0, \pi) \end{array} \right]$$

since eqn (2) holds for $k=0$

$\therefore$ By taking eqn (2) for the ball $B(x, \frac{\pi}{2})$, we have

$$|u(x)| \leq \frac{C_0}{\left(\frac{\gamma}{2}\right)^n} \|u\|_{L^1(B(x, \frac{\gamma}{2}))}$$

$$= \frac{C_0 2^n}{\gamma^n} \int_{B(x, \frac{\gamma}{2})} |u| dy$$

$$\left[\begin{array}{l} \because \|u\|_{L^p(U)} = \left(\int_U |u|^p dx \right)^{1/p} \\ \|u\|_{L^1(B(x, \frac{\gamma}{2}))} = \int_{B(x, \frac{\gamma}{2})} |u| dy \end{array} \right]$$

$$\leq \frac{C_0 2^n}{\gamma^n} \int_{B(x_0, \gamma)} |u| dy$$

using (3)

$$= \frac{C_0 2^n}{\gamma^n} \|u\|_{L^1(B(x_0, \gamma))}$$

$$= \frac{2^n}{\alpha(n) \gamma^n} \|u\|_{L^1(B(x_0, \gamma))}$$

$$[\because C_0 = \frac{1}{\alpha(n)}]$$

$$\text{Thus } |u(x)| \leq \frac{2^n}{\alpha(n) \gamma^n} \|u\|_{L^1(B(x_0, \gamma))} \quad \forall x \in B(x_0, \gamma/2)$$

$$\Rightarrow \|u\|_{L^\infty(B(x_0, \gamma/2))} \leq \frac{2^n}{\alpha(n) \gamma^n} \|u\|_{L^1(B(x_0, \gamma))} \quad \text{--- (4)}$$

$$\left[\begin{array}{l} \because \text{If } |u(x)| \leq M \quad \forall x \in U \\ \text{then } \|u\|_{L^1(U)} \leq M \end{array} \right]$$

$\therefore$ By eqn (3) & (4) we have

$$|u_{n,j}(x_0)| \leq \frac{2^n}{\gamma} \cdot \frac{2^n}{\alpha(n) \gamma^n} \|u\|_{L^1(B(x_0, \gamma))} \quad \forall j \{1 \leq j \leq n\}$$

$$\Rightarrow |D^\alpha u(x_0)| \leq \frac{C_1}{\gamma^{n+1}} \|u\|_{L^1(B(x_0, \gamma))}$$

$$\text{where } C_1 = \left(\frac{2^{n+1} n! j!}{\alpha(n)} \right) = \frac{2^{n+1} n!}{\alpha(n)}$$

for multiindex α of order 1

Thus result is true for $k=1$

Now Let $k \geq 2$ and eqn (7) is true for each ball in U and each multiindex α of order $\leq k-1$

R.T.P.: result (7) is true for each multiindex α of order k

Fix $B(x_0, r) \subset U$ and multiindex α such that $|\alpha| = k$

Then $D^\alpha u = (D^\beta u)_{x_i}$, for some $i \in \{1, 2, \dots, n\}$
 $\& |\beta| = k-1$

i.e. $D^\beta u$ is one of partial derivative of u of order $k-1$

On differentiating one more time $D^\alpha u$ is the derivative of u of order k .

Since u is harmonic

$\therefore$ all of its partial derivatives are also harmonic

$\therefore D^\beta u$ is harmonic

By induction

$$|D^\beta u(x_0)| \leq \frac{C_{k-1}}{r^{n+k-1}} \|u\|_{L^1(B(x_0, r))} \quad \text{--- (4)}$$

As $D^\beta u$ is harmonic in U

$\therefore (D^\beta u)_{x_i}$ is harmonic in U .

$\therefore$ By Mean value formula for Laplace eqn, we have

$$|(D^\beta u)_{x_i}(x_0)| = \left| \int_{B(x_0, \frac{r}{k})} (D^\beta u)_{x_i}(x) dx \right|$$

where $B(x_0, \frac{\eta}{k}) \subset B(x_0, \eta) \subset U$

(this is possible, $\because k \geq 2$)

$$\Rightarrow |D^\alpha u(x_0)| = \left| \frac{1}{\alpha(n) \cdot \left(\frac{\eta}{k}\right)^n} \int_{B(x_0, \frac{\eta}{k})} (D^\beta u)_{x_i}(x) dx \right|$$

$$= \left| \frac{k^n}{\alpha(n) \eta^n} \int_{\partial B(x_0, \frac{\eta}{k})} D^\beta u \cdot \nu_i ds \right|$$

(By Gauss

Green's theorem)

$$\leq \frac{k^n}{\alpha(n) \eta^n} \int_{\partial B(x_0, \frac{\eta}{k})} |D^\beta u| ds$$

$\left[\begin{array}{l} \because \nu \text{ is a unit vector} \\ \therefore |\nu| = 1 \end{array} \right]$

$$= \frac{k^n}{\alpha(n) \eta^n} \cdot \frac{n \alpha(n) \cdot \eta^{n-1}}{k^{n-1}} \int_{\partial B(x_0, \frac{\eta}{k})} |D^\beta u| ds$$

$$= \frac{n k}{\eta} \int_{\partial B(x_0, \frac{\eta}{k})} |D^\beta u| ds$$

$$\leq \frac{k \cdot n}{\eta} \|D^\beta u\|_{L^\infty(\partial B(x_0, \frac{\eta}{k}))}$$

$\left[\begin{array}{l} \because \text{if } |u(x)| \leq M \quad \forall x \in U, M \in \mathbb{R} \\ \text{then } |u(x)| \leq \|u\|_{L^\infty(U)} \quad \forall x \in U \\ \Rightarrow \int_U |u(x)| dx \leq \|u\|_{L^\infty(U)} \end{array} \right]$

(i.e. average of $|u|$ will also
less than $\|u\|_{L^\infty(U)}$)

$$\Rightarrow |D^\alpha u(x_0)| \leq \frac{k n}{\eta} \|D^\beta u\|_{L^\infty(\partial B(x_0, \frac{\eta}{k}))} \quad \text{--- (5)}$$

We claim that if $x \in \partial B(x_0, \frac{\eta}{k})$ $\leq \frac{C_{k+1} \cdot k^{n+k-1}}{\eta^{n+k-1} \cdot (k \eta)^{n+k-1}} \|u\|_{L^1(B(x_0, \eta))}$
if $x \in \partial B(x_0, \frac{\eta}{k})$

then $B(x, \frac{(k+1)\eta}{k}) \subseteq B(x_0, \eta) \subset U$

Let $y \in B(x, \frac{(k+1)\eta}{k})$
then $d(y, x) \leq \frac{(k+1)\eta}{k}$
consider $d(y, x_0) \leq d(y, x) + d(x, x_0)$
 $\leq \frac{(k+1)\eta}{k} + \frac{\eta}{k} = \eta$
i.e. $d(y, x_0) \leq \eta$
 $\Rightarrow y \in B(x_0, \eta)$
 $\Rightarrow B(x, \frac{(k+1)\eta}{k}) \subseteq B(x_0, \eta) \subset U$

since the result holds for $k-1$

$\therefore$ by eqⁿ (4), we have

$$|D^p u(x)| \leq \frac{C_{k-1}}{(\frac{(k+1)\eta}{k})^{n+k-1}} \|u\|_{L^1(B(x, \frac{(k+1)\eta}{k}))} \quad \forall x \in \partial B(x_0, \frac{\eta}{k})$$

$$\Rightarrow \|D^p u\|_{L^\infty(\partial B(x_0, \frac{\eta}{k}))} \leq \frac{C_{k-1} \cdot k^{n+k-1}}{(k+1)^{n+k-1} \cdot \eta^{n+k-1}} \|u\|_{L^1(B(x, \frac{(k+1)\eta}{k}))}$$

[$\because$ if $|u(x)| \leq M \quad \forall x \in U$
 $\forall M \in \mathbb{R}$
then $\|u\|_{L^\infty(U)} \leq M$]

$$\Rightarrow \|D^p u\|_{L^\infty(\partial B(x_0, \frac{\eta}{k}))} \leq \frac{C_{k-1} \cdot k^{n+k-1}}{(k+1)^{n+k-1} \cdot \eta^{n+k-1}} \|u\|_{L^1(B(x_0, \eta))} \quad \text{--- (6)}$$

$$[\because B(x, \frac{(k+1)\eta}{k}) \subseteq B(x_0, \eta)]$$

from (5) & (6), we have

$$|D^\alpha u(x_0)| \leq \frac{k\eta}{\eta} \cdot \frac{C_{k-1} k^{n+k-1}}{(k+1)^{n+k-1} \cdot \eta^{n+k-1}} \|u\|_{L^1(B(x_0, \eta))} \quad \text{--- (7)}$$

Consider, $\frac{kn}{n} \cdot \frac{c_{k-1} \cdot k^{n+k-1}}{(k-1)^{n+k-1} n^{n+k-1}} = \frac{k^{n+k} \cdot n \cdot c_{k-1}}{n^{n+k} (k-1)^{n+k-1}}$

$$= \frac{k^{n+k} \cdot n}{n^{n+k} (k-1)^{n+k-1}} \cdot \frac{(2^{n+1} \cdot n \cdot (k-1))^{k-1}}{\alpha(n)}$$

$$= \frac{k^{n+k} \cdot n}{n^{n+k} (k-1)^{n+k-1}} \cdot \frac{c_k}{c_k} \cdot \frac{(2^{n+1} \cdot n \cdot (k-1))^{k-1}}{\alpha(n)} \quad \left[\because c_k = \frac{(2^{n+1} \cdot n \cdot k)^k}{\alpha(n)} \right]$$

$$= \frac{k^{n+k} \cdot n}{n^{n+k} (k-1)^{n+k-1}} \cdot \frac{c_k}{(2^{n+1} \cdot n \cdot k)^k} \cdot \frac{(2^{n+1} \cdot n \cdot (k-1))^{k-1}}{\alpha(n)}$$

$$= \frac{c_k}{n^{n+k}} \cdot \frac{k^n \cdot k^k \cdot n}{(k-1)^n (k-1)^{k-1}} \cdot \frac{(2^{n+1})^{k-1} \cdot n^{k-1} (k-1)^{k-1}}{(2^{n+1})^k \cdot n^k \cdot k^k}$$

$$= \frac{c_k}{n^{n+k}} \cdot \frac{k^n}{(k-1)^n} \cdot \frac{1}{2^{n+1}}$$

$$= \frac{c_k}{n^{n+k}} \cdot \frac{1}{2} \cdot \left(\frac{k}{2(k-1)} \right)^n \quad \left[\begin{array}{l} k \geq 2 \Rightarrow \frac{k}{2} \geq 1 \\ \text{and } k-1 \geq 1 \Rightarrow \\ \frac{k}{2(k-1)} \leq 1 \\ \Rightarrow \left(\frac{k}{2(k-1)} \right)^n \leq 1 \\ \Rightarrow \frac{1}{2} \left(\frac{k}{2(k-1)} \right)^n < 1 \end{array} \right]$$

from eqn (7) and (8)

$$|D^\alpha u(x_0)| < \frac{C_k}{n^{n+k}} \|u\|_{L^1(B(x_0, r))}$$

Hence by principle of mathematical induction the result is true for all k .

* Some useful notations:-

1. $C_c(U)$ denote set of those functions in $C(U)$, which have compact support.

• $C_c^k(U)$ denotes set of those functions in $C^k(U)$, which have compact support.

$$(\text{supp } f = \{x: f(x) \neq 0\})$$

i.e. $C_c(U) = \{u \in C(U) \mid \overline{\{x: u(x) \neq 0\}} \text{ is compact set}\}$

$$C_c^k(U) = \{u \in C^k(U) \mid \overline{\{x: u(x) \neq 0\}} \text{ is compact set}\}$$

2. $L^p(U) = \{u: U \rightarrow \mathbb{R} \mid \int_U |u|^p dx < \infty \text{ and } u \text{ is Lebesgue measurable}\}$

Define $\|\cdot\|_{L^p(U)}: L^p(U) \rightarrow \mathbb{R}$ by

$$\|u\|_{L^p(U)} = \left(\int_U |u|^p dx \right)^{1/p} \text{ where } 1 \leq p < \infty$$

3. $L^\infty(U) = \{u: U \rightarrow \mathbb{R} \mid u \text{ is Lebesgue measurable and } \text{ess sup}_U |u| < \infty\}$

Define $\|\cdot\|_{L^\infty(U)}: L^\infty(U) \rightarrow \mathbb{R}$ by

$$\|u\|_{L^\infty(U)} = \text{ess sup}_U |u|$$

(Poisson)

* Poisson Equation :-

A non-homogeneous form of Laplace eqn

$$\Delta u = -f$$

$$\text{if } \sum_{i=1}^n u_{x_i} x_i = -f$$

is called Poisson eqn

Here $x \in U$ and U is open subset of $\mathbb{R}^n$ $u: \bar{U} \rightarrow \mathbb{R}$ and $f \in C^0(\bar{U})$

The solution u of Poisson's eqⁿ is defined by formula

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$$

where Φ is fundamental solⁿ of Laplace eqⁿ

$$\therefore u(x) = \begin{cases} -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| f(y) dy & ; n=2 \\ \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy & ; n \geq 3 \end{cases}$$

* Liouville's Theorem:-

suppose $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is harmonic and bounded, Then prove that u is constant.

ie there doesn't exist non-trivial bounded harmonic function on $\mathbb{R}^n$

Proof:-

It is given that $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is harmonic.

$$\therefore \Delta u = 0$$

Now fix $x_0 \in \mathbb{R}^n$ and $\pi > 0$ and take ball $B(x_0, \pi) \subset \mathbb{R}^n$,

then

$$|D^2 u(x_0)| = \left(\sum_{|\alpha|=2} |D^\alpha u(x_0)|^2 \right)^{1/2} \quad \text{--- (1)}$$

where α is a multiindex.

We know from the Local Estimate theorem,

$$|D^\alpha u(x_0)| \leq \frac{C_k}{\pi^{n+k}} \|u\|_{L^1(B(x_0, \pi))}$$

for each $B(x_0, \pi) \subset U$ and multiindex α

with $|\alpha| = k$ and $k = 0, 1, 2, \dots$

$$\text{and } C_0 = \frac{1}{\alpha(n)}, \quad C_k = \frac{(2^{n+1} n^k)^k}{\alpha(n)}$$

∴ for $|\alpha| = k \geq 1$, we have

$$|\mathcal{D}^\alpha u(x_0)| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0, r))} \quad \text{--- (2)}$$

∴ By using (1) & (2), we have

$$|\mathcal{D}' u(x_0)| \leq \left[\sum_{|\alpha|=1} \left(\frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0, r))} \right)^2 \right]^{1/2}$$

$$= \left[\left(\frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0, r))} \right)^2 \sum_{|\alpha|=1} 1^2 \right]^{1/2}$$

$$= \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0, r))} \sqrt{n}$$

$$\leq \frac{C_1 \sqrt{n}}{r^{n+1}} \|u\|_{L^1(\mathbb{R}^n)}$$

$$\rightarrow 0 \text{ as } r \rightarrow \infty$$

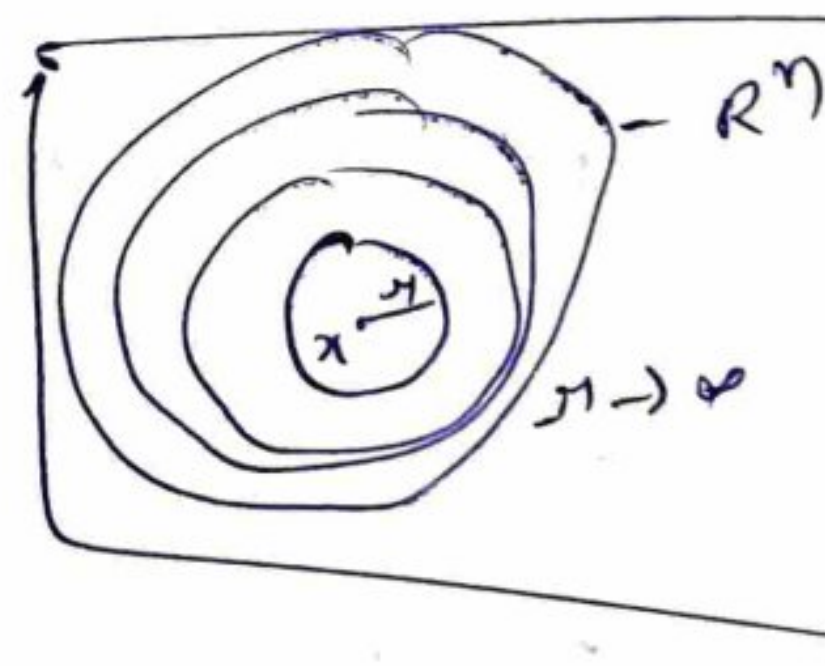
$$\Rightarrow |\mathcal{D} u(x)| \rightarrow 0 \text{ as } r \rightarrow \infty$$

i.e. first order derivative of $u \rightarrow 0$

$\Rightarrow u$ is constant.

$$\left[\begin{array}{l} \because \alpha = (\alpha_1, \dots, \alpha_n) \\ \text{either } \alpha_1 \geq 1 \text{ or } \alpha_2 \geq 1 \\ \dots \text{ or } \alpha_n \geq 1 \end{array} \right]$$

$$\left[\because B(x_0, r) \subseteq \mathbb{R}^n \right]$$



* Representation Formula:-

Let $f \in C_c^\infty(\mathbb{R}^n)$, $n \geq 3$

then any bounded solution of $\Delta u = -f$ in $\mathbb{R}^n$ has the form :-

$$u(x) = \int_{\mathbb{R}^n} \Phi(y-x) f(y) dy + C$$

for some constant C .

Solⁿ:- for $n \geq 3$, We know that fundamental solution of homogeneous Laplace eqn, i.e. $\Delta u = 0$ is

$$\Phi(x) = \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} \quad x \neq 0, x \in \mathbb{R}^n \quad (n \geq 3)$$

$\therefore$ fundamental solⁿ for Laplace eqⁿ is

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & \text{if } n=2, x \in \mathbb{R}^n, x \neq 0 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} & \text{if } n \geq 3, x \in \mathbb{R}^n, x \neq 0 \end{cases}$$

Now $f \in C_c^2(\mathbb{R}^n)$ and $n \geq 3$

since $\Phi(x) \rightarrow 0$ as $|x| \rightarrow \infty$ for $n \geq 3$

$\therefore \tilde{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$ is a bounded solution of

$$\Delta u = -f \text{ in } \mathbb{R}^n \quad \text{--- ①}$$

[By Poisson eqⁿ theorem]

Let 'u' be any other bounded solⁿ of ①

$$\text{Let } w := u - \tilde{u}$$

since u & $\tilde{u}$ are bounded.

$\therefore w$ is also bounded

Now

$$\begin{aligned} \Delta w &= \Delta(u - \tilde{u}) = \Delta u - \Delta \tilde{u} \\ &= -f + f = 0 \end{aligned}$$

$$\text{i.e. } \Delta w = 0$$

$\Rightarrow w$ satisfies Laplace eqⁿ

Thus w is harmonic function

$\therefore$ By Liouville's theorem, w is constant

i.e. $w = c$ for some constant c

$$\Rightarrow u - \tilde{u} = c$$

$$\Rightarrow u = \tilde{u} + c$$

$$\text{i.e. } u(x) = \tilde{u}(x) + c$$

$\Rightarrow$

$$\boxed{u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy + c}$$

Green Formula:-

Let $u, v \in C^2(\bar{U})$, not necessarily harmonic and satisfies poisson eqn. Then

$$\int_U (u \Delta v - v \Delta u) dx = \int_{\partial U} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) ds$$

where $\frac{\partial v}{\partial \nu} = \nabla v \cdot \nu$ is the directional derivatives of v in the direction of ν

and $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ is directional derivative of u in direction of ν

Derivation of Green's function:-

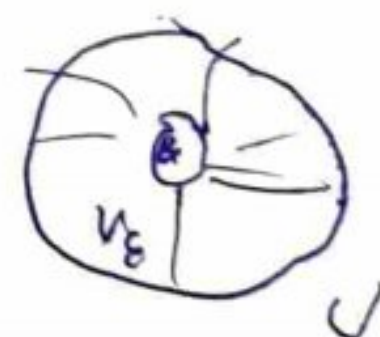
Let $U \subseteq \mathbb{R}^n$ be open and bounded. Let $u \in C^2(\bar{U})$ be any arbitrary function. Let $x \in U$ be fixed. Then find the value of $u(x)$ in terms of Green's functions.

Soln:- Let $U \subseteq \mathbb{R}^n$ be open and bounded.

for fixed $x \in U$

choose an $\epsilon > 0$ s.t. $B(x, \epsilon) \subseteq U$

$$\text{Let } V_\epsilon = U - B(x, \epsilon)$$



Let Φ be the fundamental solution of Laplace eqn. Then applying Green formula for the functions $u(y)$ and $\Phi(y-x)$ on the region V_ϵ , we get

$$\int_{V_\epsilon} (u(y) \Delta \Phi(y-x) - \Phi(y-x) \Delta u(y)) dy = \int_{\partial V_\epsilon} \left(u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu}(y) \right) ds \quad \text{--- (1)}$$

Since Φ is a fundamental soln of Laplace eqn

$$\therefore \Delta \Phi(y-x) = 0$$

∴ eqn ① implies

$$-\int_{V_\varepsilon} \Phi(y-x) \Delta u(y) dy = \int_{\partial V_\varepsilon} \left(u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u(y)}{\partial \nu} \right) ds \quad \text{--- ②}$$

since $V_\varepsilon = U - B(x, \varepsilon)$

∴ $\partial V_\varepsilon = (\partial U) \cup (\partial B(x, \varepsilon))$

Thus ② $\Rightarrow$

$$\begin{aligned} -\int_{V_\varepsilon} \Phi(y-x) \Delta u(y) dy &= \int_{\partial U} \left(u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u(y)}{\partial \nu} \right) ds \\ &\quad + \int_{\partial B(x, \varepsilon)} \left(u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u(y)}{\partial \nu} \right) ds \\ &= \int_{\partial U} \left(u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u(y)}{\partial \nu} \right) ds \\ &\quad + \int_{\partial B(x, \varepsilon)} u(y) \frac{\partial \Phi(y-x)}{\partial \nu} ds - \int_{\partial B(x, \varepsilon)} \Phi(y-x) \frac{\partial u(y)}{\partial \nu} ds \end{aligned} \quad \text{--- ③}$$

consider

$$\begin{aligned} \left| \int_{\partial B(x, \varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu} ds \right| &\leq \int_{\partial B(x, \varepsilon)} |\Phi(y-x)| \left| \frac{\partial u}{\partial \nu} \right| ds \\ &\leq \max_{\partial B(x, \varepsilon)} |\Phi| \int_{\partial B(x, \varepsilon)} |\mathcal{D}u(y)| ds \quad \{ \because |\nu|=1 \} \\ &= \max_{\partial B(x, \varepsilon)} |\Phi| \int_{\partial B(x, \varepsilon)} |\mathcal{D}u(y)| ds \\ &= \max_{\partial B(x, \varepsilon)} |\Phi| n \alpha(n) \cdot \varepsilon^{n-1} \int_{\partial B(x, \varepsilon)} |\mathcal{D}u(y)| ds \\ &\leq \max_{\partial B(x, \varepsilon)} |\Phi| \cdot n \alpha(n) \cdot \varepsilon^{n-1} \| \mathcal{D}u \|_{\infty(\partial B(x, \varepsilon))} \end{aligned}$$

$$\Rightarrow \left| \int_{\partial B(x, \varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu} ds \right| \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

$$\Rightarrow \int_{\partial B(x, \varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu} ds \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad \text{--- ④}$$

Now consider

$$\int_{\partial B(x, \epsilon)} u(y) \frac{\partial \Phi}{\partial \nu} (y-x) ds = \int_{\partial B(x, \epsilon)} u(y) (\nabla \Phi(y-x) \cdot \nu) ds$$

— (8)

Here Φ is fundamental solution of Laplace's eqn

$$\therefore \Phi(y-x) = \begin{cases} -\frac{1}{2n} \log |y-x| & \text{if } n=2 \\ \frac{1}{n(n-2)\alpha(n)} |y-x|^{2-n} & \text{if } n \geq 3 \end{cases}$$

$$\nabla \Phi(y-x) = \begin{cases} -\frac{1}{2n} \cdot \frac{1}{|y-x|} \cdot \frac{y-x}{|y-x|} & \text{if } n=2 \\ \frac{(2-n)}{n(n-2)\alpha(n)} \cdot \frac{y-x}{|y-x|^{n-1}} & \text{if } n \geq 3 \end{cases}$$

$$= \begin{cases} -\frac{(y-x)}{2n |y-x|^2} & \text{if } n=2 \\ -\frac{(y-x)}{n \alpha(n) |y-x|^n} & \text{if } n \geq 3 \end{cases}$$

$$\Rightarrow \nabla \Phi(y-x) = -\frac{(y-x)}{n \alpha(n) |y-x|^n} \text{ if } n \geq 2$$

$$\text{Let } \gamma = \frac{-(y-x)}{|y-x|}$$

$$\text{then } |\gamma| = \frac{|y-x|}{|y-x|} = 1, \text{ i.e., } \gamma \text{ is a unit vector}$$

$$\therefore \nabla \Phi(y-x) \cdot \nu = \frac{-(y-x)}{n \alpha(n) |y-x|^n} \cdot \frac{-(y-x)}{|y-x|} = \frac{|y-x|^2}{n \alpha(n) |y-x|^{n+1}}$$

$$\text{i.e. } \nabla \Phi(y-x) \cdot \nu = \frac{1}{n \alpha(n) |y-x|^{n-1}}$$

$\therefore$ (8) implies

$$\int_{\partial B(x, \epsilon)} u(y) \frac{\partial \Phi}{\partial \nu} (y-x) ds = \int_{\partial B(x, \epsilon)} u(y) \frac{1}{n \alpha(n) |y-x|^{n-1}} ds(y)$$

$$\begin{aligned} \because \nabla (\log |y-x|) &= \frac{1}{|y-x|} \cdot \frac{y-x}{|y-x|} \\ &= \frac{y-x}{|y-x|^2} \end{aligned}$$

$$\begin{aligned} \nabla \left[\frac{1}{(y-x)^{n-2}} \right] &= \frac{(2-n)}{|y-x|^{n-1}} \cdot \frac{y-x}{|y-x|} \\ &= \frac{2-n}{|y-x|^n} \end{aligned}$$

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right)$$

$$\begin{aligned} \nabla(|x|) &= \left(\frac{\partial |x|}{\partial x_1}, \dots, \frac{\partial |x|}{\partial x_n} \right) \\ &= \frac{x_1}{|x|}, \dots, \frac{x_n}{|x|} \\ &= \frac{1}{|x|} (x_1, \dots, x_n) = \frac{x}{|x|} \end{aligned}$$

$x \neq 0$

$$\because x \cdot x = x_1^2 + x_2^2 + \dots + x_n^2 = |x|^2$$

$$\Rightarrow x \cdot x = |x|^2$$

$$\begin{aligned}
 &= \int_{\partial B(x, \epsilon)} u(y) \cdot \frac{1}{n \cdot \alpha(n) \cdot \epsilon^{n-1}} ds(y) \\
 &= \frac{1}{n \cdot \alpha(n) \cdot \epsilon^{n-1}} \int_{\partial B(x, \epsilon)} u(y) \cdot ds(y) \quad \left[\begin{array}{l} U \subset \partial B(x, \epsilon) \\ \Rightarrow d(y, x) < \epsilon \\ \Rightarrow |y - x| < \epsilon^{n-1} \end{array} \right] \\
 &= \int_{\partial B(x, \epsilon)} u(y) ds(y) \rightarrow u(x) \text{ as } \epsilon \rightarrow 0
 \end{aligned}$$

(47)

using eqn (3), (4) & (5) we get

$$\begin{aligned}
 - \lim_{\epsilon \rightarrow 0} \int_{V_\epsilon} \Phi(y-x) \Delta u(y) dy &= \int_{\partial U} \left(u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu} \right) ds \\
 &\quad + \lim_{\epsilon \rightarrow 0} \int_{\partial B(x, \epsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y-x) ds - \lim_{\epsilon \rightarrow 0} \int_{\partial B(x, \epsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu} ds \\
 &= \int_{\partial U} \left(u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu} \right) ds + u(x) + 0
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow u(x) &= - \int_{\partial U} \left(u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu} \right) ds + \lim_{\epsilon \rightarrow 0} \int_{V_\epsilon} \Phi(y-x) \Delta u(y) dy \\
 &= \int_{\partial U} \left(\Phi(y-x) \frac{\partial u}{\partial \nu} - u(y) \frac{\partial \Phi}{\partial \nu}(y-x) \right) ds = \int_U \Phi(y-x) \Delta u(y) dy \quad \text{--- (7)}
 \end{aligned}$$

Now to determine $u(x)$, we need Δu in U and u , $\frac{\partial u}{\partial \nu}$ along ∂U . But all these functions are unknown, so to avoid these terms, we introduce corrector function ϕ^x (for fixed x) as follows :-

$$\phi^x = \phi^x(y)$$

$$\left. \begin{array}{l} \text{and } \Delta \phi^x(y) = 0 \text{ in } U \\ \text{and } \phi^x(y) = \Phi(y-x) \text{ on } \partial U \end{array} \right\} \quad \text{--- (8)}$$

Now applying Green's formula to $\phi^x(y)$ and $u(y)$ in U , we get

$$\int_U (\phi^x(y) \Delta u(y) - u(y) \Delta \phi^x(y)) dy$$

$$= \int_{\partial U} \left(\phi^x(y) \frac{\partial u}{\partial \nu} - u(y) \frac{\partial \phi^x}{\partial \nu} \right) ds$$

Since $\Delta \phi^x(y) = 0$ in U

$\therefore$ we get, $\int_U \phi^x(y) \Delta u(y) dy = \int_{\partial U} \left(\phi^x(y) \frac{\partial u}{\partial \nu} - u(y) \frac{\partial \phi^x}{\partial \nu} \right) ds$

$$0 = - \int_U \phi^x(y) \Delta u(y) dy + \int_{\partial U} \left(\phi^x(y) \frac{\partial u}{\partial \nu} - u(y) \frac{\partial \phi^x}{\partial \nu} \right) ds$$

subtracting ⑦ & ⑧, we get

$$u(x) = \int_{\partial U} \left(\Phi(y-x) \frac{\partial u}{\partial \nu} - u(y) \frac{\partial \Phi}{\partial \nu}(y-x) \right) ds$$

$$- \int_U \Phi(y-x) \Delta u(y) dy$$

$$+ \int_U \Phi^x(y) \Delta u(y) dy - \int_{\partial U} \left(\phi^x(y) \frac{\partial u}{\partial \nu} - u(y) \frac{\partial \phi^x}{\partial \nu} \right) ds$$

But $\Phi(y-x) = \phi^x(y)$ on ∂U

$$u(x) = \int_{\partial U} \cancel{\Phi(y-x)} \frac{\partial u}{\partial \nu} ds - \int_{\partial U} u(y) \frac{\partial \Phi}{\partial \nu}(y-x) ds - \int_{\partial U} \cancel{\phi^x(y)} \frac{\partial u}{\partial \nu} ds$$

$$+ \int_{\partial U} u(y) \frac{\partial \phi^x}{\partial \nu}(y) ds$$

$$+ \int_U (\phi^x(y) - \Phi(y-x)) \Delta u(y) dy$$

$$u(x) = - \int_{\partial U} \frac{\partial}{\partial \nu} (\Phi(y-x) - \phi^x(y)) u(y) ds(y)$$

$$- \int_U (\Phi(y-x) - \phi^x(y)) \Delta u(y) dy$$

Now we define Green's function as follows: (48)

~~Now~~ we define Green's function as

$$G(x, y) = \Phi(y-x) - \phi^x(y) \quad \text{for fixed } x \in U$$

$$u(x) = - \int_{\partial U} \frac{\partial G(x, y)}{\partial \nu} u(y) ds(y) - \int_U G(x, y) \Delta u(y) dy$$

(Corrector Function): Let $U \subseteq \mathbb{R}^n$ be open and bounded for $x \in U$, we define a corrector function

$$\phi^x = \phi^x(y) \quad \text{which is soln of B.V.P.}$$

$$\Delta \phi^x(y) = 0 \quad \text{in } U$$

and $\phi^x(y) = \Phi(y-x) \quad \text{on } \partial U$

Here Φ is a fundamental solution of Laplace equation. Here $y \in U$ is a variable and $x \in U$ is fixed.

** Green Function :- Let $U \subseteq \mathbb{R}^n$ be open and bounded. For fixed $x \in U$, we define Green Function

$$G(x, y) \quad \text{as} \quad G(x, y) = \Phi(y-x) - \phi^x(y)$$

where Φ is fundamental soln of Laplace eqn and ϕ^x is the corrector function. Here y is variable and x is fixed.

** Representation formula in terms of Green's functions:-

i.e

The soln of the poisson eqn with boundary condition in terms of Green's function

Theorem:- Let $u \in C^2(\bar{U})$ and $U \subseteq \mathbb{R}^n$ be open and bounded. (50)
 Let u be a solⁿ of B.V.P

$$\text{and } \left. \begin{array}{l} \Delta u = -f \quad \text{in } U \\ u = g \quad \text{on } \partial U \end{array} \right\} \text{--- } (*)$$

where f and g are continuous functions.

Then,

$$u(x) = - \int_{\partial U} \frac{\partial G(x,y)}{\partial \nu} \cdot g(y) ds(y) + \int_U G(x,y) f(y) dy \quad (\text{where } x \in U)$$

Proof:- since $u \in C^2(\bar{U})$

$\therefore$ By last article

$$u(x) = - \int_{\partial U} \frac{\partial G(x,y)}{\partial \nu} \cdot u(y) ds(y) - \int_U G(x,y) \Delta u(y) dy \text{--- } (1)$$

(Prove it)

since u is a solution of $(*)$

$$\therefore \Delta u = -f \text{ in } U$$

$$u = g \text{ on } \partial U$$

put these value in eqⁿ (1)

$$u(x) = - \int_{\partial U} \frac{\partial G(x,y)}{\partial \nu} g(y) ds(y) + \int_U G(x,y) f(y) dy$$

Symmetry of Green's functions:-

Theorem:- Let $U \subseteq \mathbb{R}^n$ be open and bounded. Then
 Prove that $G(x,y) = G(y,x) \quad \forall \quad x,y \in U$ and
 $x \neq y$ where $G(x,y) = \Phi(y-x) - \Phi^x(y)$

Proof:- fix $x, y \in U$ be such that $x \neq y$

and define $v(z) = G(x, z) = \Phi(z-x) - \phi^x(z)$

and $w(z) = G(y, z) = \Phi(z-y) - \phi^y(z)$
where $z \in U$

where ϕ^x is the corrector function and

Φ is the fundamental solⁿ of Laplace eqⁿ

(we have to prove)

R.T.P :-

$$G(x, y) = G(y, x)$$

i.e. we have to prove $v(y) = w(x)$

$$\text{Now } \Delta v(z) = \Delta \Phi(z-x) - \Delta \phi^x(z) \quad \text{where } z \neq x$$

$$= 0 - 0$$

$$= 0 \text{ in } U$$

$$\text{similarly } \Delta w(z) = \Delta \Phi(z-y) - \Delta \phi^y(z)$$

$$= 0 - 0$$

$$= 0 \text{ in } U$$

$\because \Phi$ is fundamental solⁿ of Laplace eqⁿ $\therefore \Delta \Phi = 0$

Also $\phi^x(y)$ is corrector function $\therefore \Delta \phi^x(y) = 0$ (by defⁿ)

$$\therefore \Delta \Phi = 0 \neq \Delta \phi^y(z) = 0$$

$$\begin{aligned} \text{Now on } \partial U, \quad v(z) &= \Phi(z-x) - \phi^x(z) \\ &= \Phi(z-x) - \Phi(z-x) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{and } w(z) &= \Phi(z-y) - \phi^y(z) \\ &= \Phi(z-y) - \Phi(z-y) \\ &= 0 \end{aligned}$$

$\therefore$ by defⁿ of corrector function $\phi^x(y) = \Phi(y-x)$ on ∂U where Φ is fundamental solⁿ of Laplace eqⁿ

$$\therefore \Delta v(z) = \Delta w(z) \text{ in } U \text{ and}$$

$$\text{and } v(z) = w(z) \text{ on } \partial U$$

choose sufficiently small $\epsilon > 0$ and $V = U - (B(x, \epsilon) \cup B(y, \epsilon))$

Since $\Delta v = 0$ and $\Delta w = 0$ in U

$$\therefore v, w \in C^2(U)$$

$\therefore$ applying Green's formula to w and v in the region V .

$$\int_V (\omega \Delta v - v \Delta \omega) dz = \int_{\partial V} \left(\omega \frac{\partial v}{\partial \bar{v}} - v \frac{\partial \omega}{\partial \bar{v}} \right) ds \quad \text{--- (1)}$$

since $\Delta v(z) = 0 = \Delta \omega(z)$ in U and $V \subseteq U$

$$\therefore \Delta v(z) = 0 = \Delta \omega(z) \text{ in } V$$

put these values in eqn (1)

$$0 = \int_{\partial V} \left(\omega \frac{\partial v}{\partial \bar{v}} - v \frac{\partial \omega}{\partial \bar{v}} \right) ds(z) \quad \text{--- (2)}$$

$$\text{since } V = U - (B(x, \epsilon) \cup B(y, \epsilon))$$

$$\therefore \partial V = \partial U \cup \partial B(x, \epsilon) \cup \partial B(y, \epsilon)$$

$\therefore$ eqn (2) implies

$$\int_{\partial U} \left(\omega \frac{\partial v}{\partial \bar{v}} - v \frac{\partial \omega}{\partial \bar{v}} \right) ds(z) + \int_{\partial B(x, \epsilon)} \left(\omega \frac{\partial v}{\partial \bar{v}} - v \frac{\partial \omega}{\partial \bar{v}} \right) ds(z) + \int_{\partial B(y, \epsilon)} \left(\omega \frac{\partial v}{\partial \bar{v}} - v \frac{\partial \omega}{\partial \bar{v}} \right) ds(z) = 0 \quad \text{--- (3)}$$

$$\text{since } v(z) = 0 = \omega(z) \text{ on } \partial U$$

$$\therefore \int_{\partial U} \left(\omega \frac{\partial v}{\partial \bar{v}} - v \frac{\partial \omega}{\partial \bar{v}} \right) ds(z) = 0$$

$$\therefore (3) \Rightarrow \int_{\partial B(x, \epsilon)} \left(\omega \frac{\partial v}{\partial \bar{v}} - v \frac{\partial \omega}{\partial \bar{v}} \right) ds(z) = \int_{\partial B(y, \epsilon)} \left(v \frac{\partial \omega}{\partial \bar{v}} - \omega \frac{\partial v}{\partial \bar{v}} \right) ds(z)$$

Taking limit $\epsilon \rightarrow 0$ both sides we get

$$\lim_{\epsilon \rightarrow 0} \int_{\partial B(x, \epsilon)} \left(\omega \frac{\partial v}{\partial \bar{v}} - v \frac{\partial \omega}{\partial \bar{v}} \right) ds(z) = \lim_{\epsilon \rightarrow 0} \int_{\partial B(y, \epsilon)} \left(v \frac{\partial \omega}{\partial \bar{v}} - \omega \frac{\partial v}{\partial \bar{v}} \right) ds(z) \quad \text{--- (4)}$$

$$\text{Now consider } \left| \int_{\partial B(x, \epsilon)} v \frac{\partial \omega}{\partial \bar{v}} ds(z) \right| \leq \int_{\partial B(x, \epsilon)} \left| v \frac{\partial \omega}{\partial \bar{v}} \right| ds(z)$$

$$= \int_{\partial B(x, \varepsilon)} |v| \left| \frac{\partial w}{\partial \nu} \right| ds(z)$$

$$\leq \max_{\partial B(x, \varepsilon)} |v| \int_{\partial B(x, \varepsilon)} |\mathcal{D}w \cdot \nu| ds(z) \quad \left[\because \frac{\partial w}{\partial \nu} = \mathcal{D}w \cdot \nu \right]$$

$$= \max_{\partial B(x, \varepsilon)} |v| \int_{\partial B(x, \varepsilon)} |\mathcal{D}w| ds(z)$$

$$= \max_{\partial B(x, \varepsilon)} |v| \cdot n \alpha(n) \varepsilon^{n-1} \int_{\partial B(x, \varepsilon)} |\mathcal{D}w| ds(z)$$

$$\leq \max_{\partial B(x, \varepsilon)} |v| \cdot n \alpha(n) \varepsilon^{n-1} \|\mathcal{D}w\|_{\infty}(\partial B(x, \varepsilon))$$

$$\rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

$$\Rightarrow \left| \int_{\partial B(x, \varepsilon)} v \frac{\partial w}{\partial \nu} ds(z) \right| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

$$\Rightarrow \int_{\partial B(x, \varepsilon)} \frac{\partial w}{\partial \nu} ds(z) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

$$\therefore \lim_{\varepsilon \rightarrow 0} \int_{\partial B(x, \varepsilon)} v \frac{\partial w}{\partial \nu} ds(z) = 0 \quad \text{--- (5)}$$

$$\begin{aligned} \text{Now consider } \int_{\partial B(x, \varepsilon)} w \cdot \frac{\partial v}{\partial \nu} ds(z) &= \int_{\partial B(x, \varepsilon)} w(z) \left[\frac{\partial}{\partial \nu} (\Phi(z-x) - \phi^x(z)) \right] ds(z) \\ &= \int_{\partial B(x, \varepsilon)} w(z) \frac{\partial \Phi}{\partial \nu}(z-x) ds(z) - \int_{\partial B(x, \varepsilon)} w(z) \cdot \frac{\partial \phi^x}{\partial \nu}(z) ds(z) \quad \text{--- (6)} \end{aligned}$$

$$\text{Since } \int_{\partial B(x, \varepsilon)} w(z) \frac{\partial \phi^x}{\partial \nu}(z) ds(z) = 0$$

$$\begin{aligned} \therefore (6) \Rightarrow \int_{\partial B(x, \varepsilon)} w \cdot \frac{\partial v}{\partial \nu} ds(z) &= \int_{\partial B(x, \varepsilon)} w(z) \frac{\partial \Phi}{\partial \nu}(z-x) ds(z) \\ &= \int_{\partial B(x, \varepsilon)} w(z) [\mathcal{D}\Phi(z-x) \cdot \nu] ds(z) \\ &= \int_{\partial B(x, \varepsilon)} w(z) \frac{1}{n \alpha(n) |z-x|^{n-1}} ds(z) \end{aligned}$$

(54)

$$\begin{aligned} \therefore \Phi(x) &= \begin{cases} -\frac{1}{2n} \log |x| & \text{if } n=2 \\ \frac{1}{n(n-2)\alpha(n)} |x|^{n-2} & \text{if } n \geq 3 \end{cases} \\ \therefore \mathcal{D}\Phi(x) &= \begin{cases} -\frac{x}{2n|x|^2} & \text{if } n=2 \\ -\frac{x}{n\alpha(n)} |x|^{n-1} & \text{if } n \geq 3 \end{cases} \\ \text{Let } v &= -\frac{x}{|x|} \\ \therefore \mathcal{D}\Phi(x) \cdot v &= \frac{x^2}{n\alpha(n) \cdot |x|^{n+1}} \\ &= \frac{1}{n\alpha(n) |x|^{n-1}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_{\partial B(x, \varepsilon)} w(z) \frac{\partial v}{\partial \nu} dS(z) &= \int_{\partial B(x, \varepsilon)} \frac{w(z)}{n\alpha(n) \varepsilon^{n-1}} dS(z) \quad \left[\begin{array}{l} \because \text{on boundary } \partial B(x, \varepsilon) \\ \text{if } z \in \partial B(x, \varepsilon) \\ \Rightarrow |z-x| = \varepsilon \end{array} \right] \\ &= \frac{1}{n\alpha(n) \varepsilon^{n-1}} \int_{\partial B(x, \varepsilon)} w(z) dS(z) \\ &= \int_{\partial B(x, \varepsilon)} w(z) dS(z) \end{aligned}$$

$$\longrightarrow w(x) \quad \text{as } \varepsilon \rightarrow 0$$

$$\therefore \lim_{\varepsilon \rightarrow 0} \int_{\partial B(x, \varepsilon)} w(z) \frac{\partial v}{\partial \nu} dS(z) = w(x) \quad \text{--- (7)}$$

using (6) & (7) we get from L.H.S of (4)

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B(x, \varepsilon)} \left(w \frac{\partial v}{\partial \nu} - v \frac{\partial w}{\partial \nu} \right) dS(z) = w(x) - 0 = w(x) \quad \text{--- (8)}$$

i.e. R.H.S of (4) similarly by symmetry

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B(y, \varepsilon)} \left(v \frac{\partial w}{\partial \nu} - w \frac{\partial v}{\partial \nu} \right) dS(z) = v(y) \quad \text{--- (9)}$$

Now using (4), (8) and (9), we get

$$w(x) = v(y)$$

$$G(y, x) = G(x, y)$$

$\Rightarrow$

i.e. G is symmetric

* Defⁿ:- (Half space R_+^n) :-

Half space R_+^n is defined as

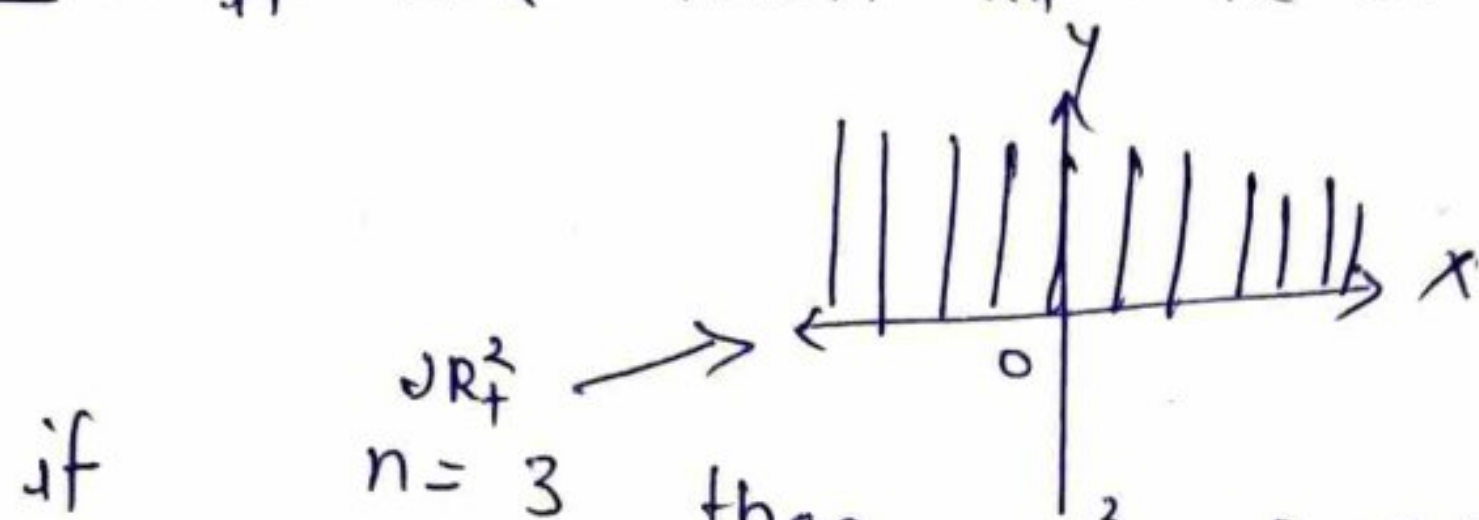
$$R_+^n = \{x = (x_1, x_2, \dots, x_n) \in R^n \mid x_n > 0\}$$

This region is unbounded and boundary of R_+^n ,

$$\text{i.e. } \partial R_+^n = \{x = (x_1, x_2, \dots, x_n) \in R^n \mid x_n = 0\}$$

Example :- if $n=2$ then $R_+^2 = \{(x, y) \in R^2 \mid y > 0\}$

$$\partial R_+^2 = \{(x, y) \in R^2 \mid y = 0\}$$

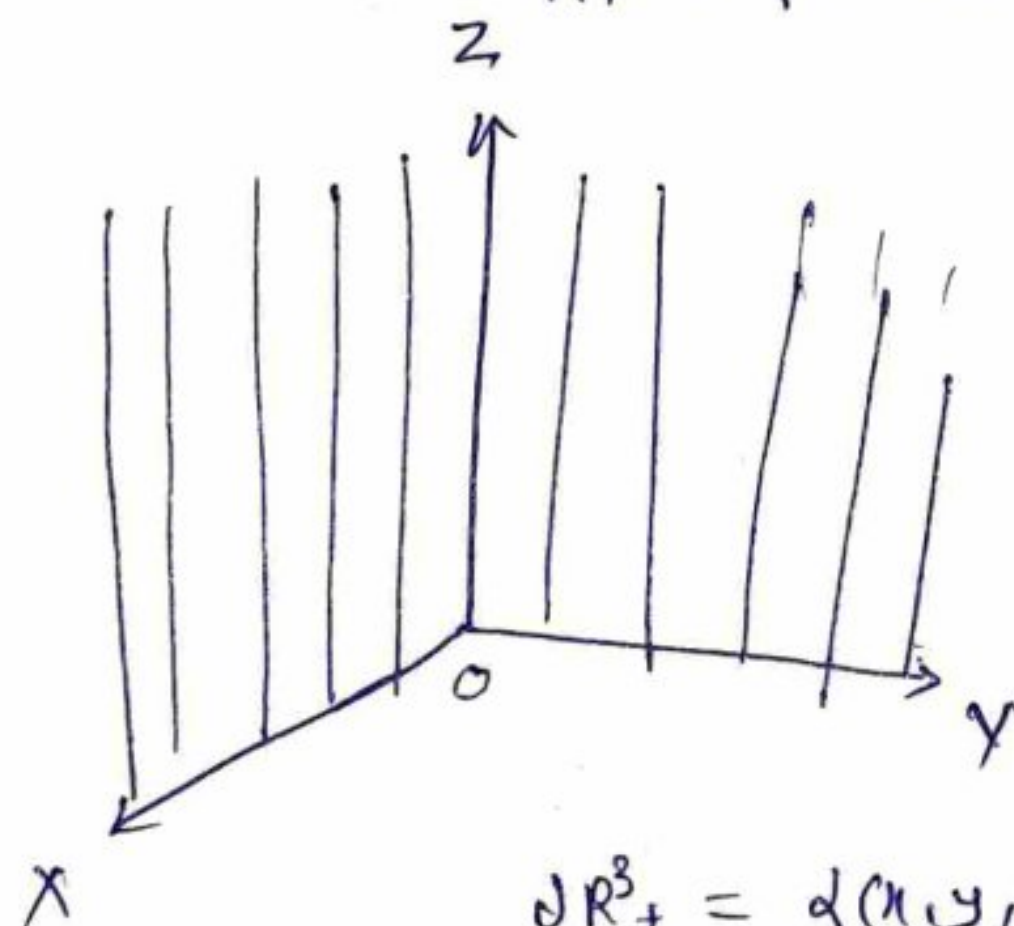


∂R_+^2

$n=2$

then

$$R_+^3 = \{(x, y, z) \in R^3 \mid z > 0\}$$



$$\partial R_+^3 = \{(x, y, z) \in R^3 \mid z = 0\}$$

* Defⁿ:- (Reflection of $x \in R_+^n$)

If $x = (x_1, x_2, \dots, x_n) \in R_+^n$ then its reflection in the plane ∂R_+^n is the point

$$\tilde{x} = (x_1, x_2, \dots, x_{n-1}, -x_n), \quad x_n > 0$$

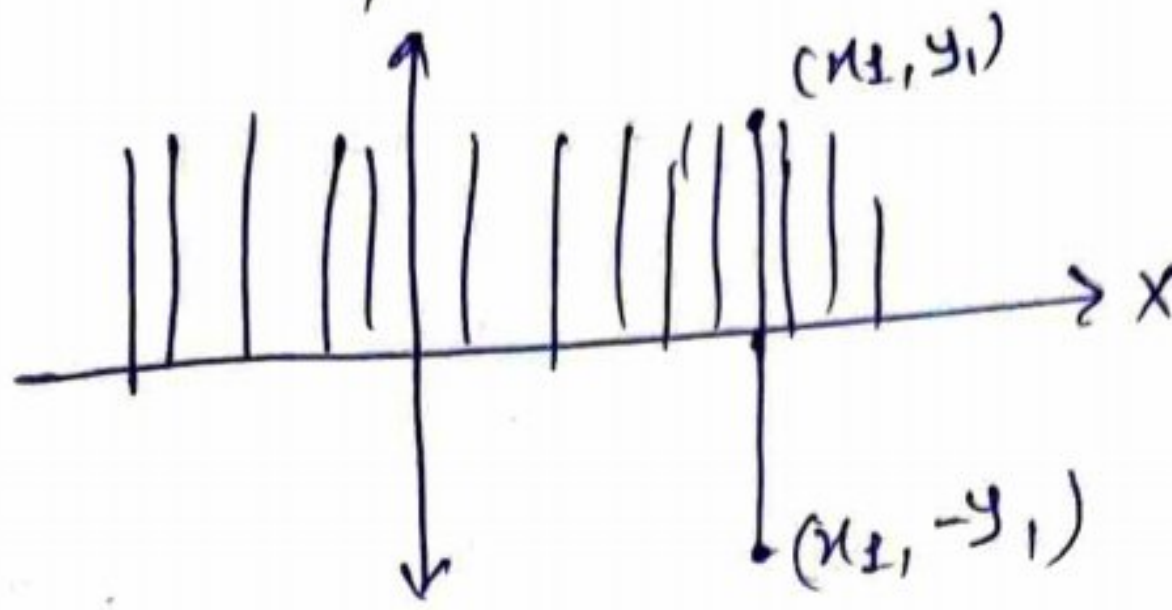
Clearly $\tilde{x} \notin R_+^n$

[$\therefore$ for fixed x , we introduce corrector function $\phi^x = \phi^x(y)$, solving a B.V.P. $\Delta \phi^x = 0$ in U $\phi^x(y) = \Phi(y-x)$ on ∂U for $y \in U$] $\rightarrow \otimes$

(Example for reflection)

Example

$$R_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$$



Now the idea is that the corrector function ϕ^n is built from Φ by "reflecting the singularity from $x \in R_+^n$ to $\tilde{x} \in R_+^n$

we define a function

$$\begin{aligned} \phi^n(y) &= \Phi(y - \tilde{x}), \text{ where } x, y \in R_+^n \\ &= \Phi(y_1 - x_1, y_2 - x_2, \dots, y_n + x_n) \end{aligned} \quad \text{--- (1)}$$

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & , n=2 \\ \frac{1}{n(n-2)\alpha(n)} |x|^{n-2} & , n \geq 3 \end{cases} \quad \text{--- (2)}$$

Since $\Phi(y - \tilde{x})$ has singularity

$\Rightarrow \Phi(y - \tilde{x})$ has no singularity in R_+^n at $y = \tilde{x}$, but $\tilde{x} \notin R_+^n$

$\Rightarrow \phi^n(y)$ has no singularity in R_+^n using (2)

$$\Phi(y - x) = \begin{cases} -\frac{1}{2\pi} \log |y - x| & n=2 \\ \frac{1}{n(n-2)\alpha(n)} |y - x|^{n-2} & n \geq 3 \end{cases}$$

$\because$ at $y = \tilde{x}$

$$\begin{aligned} \Phi(y - \tilde{x}) &= \Phi(0) \\ \text{and } \Phi(0) &= \infty \\ \therefore 0 &\text{ is singularity} \end{aligned}$$

$n=2$ --- (3)

If $y = (y_1, y_2, \dots, y_n) \in \partial R_+^n$, then $y_n = 0$

i.e. $y = (y_1, y_2, \dots, y_{n-1}, 0)$

$$\begin{aligned} y - \tilde{x} &= (y_1, y_2, \dots, y_{n-1}, 0) - (x_1, x_2, \dots, x_{n-1}, -x_n) \\ &= (y_1 - x_1, y_2 - x_2, \dots, y_{n-1} - x_{n-1}, x_n) \end{aligned}$$

and $y - x = (y_1 - x_1, y_2 - x_2, \dots, y_{n-1} - x_{n-1}, -x_n)$

$$\Rightarrow |y-\tilde{x}| = \left((y_1-x_1)^2 + (y_2-x_2)^2 + \dots + x_n^2 \right)^{1/2}$$

$$|y-x| = \left((y_1-x_1)^2 + (y_2-x_2)^2 + \dots + x_n^2 \right)^{1/2}$$

$$\therefore |y-\tilde{x}| = |y-x| \quad \text{--- (7)}$$

$$\therefore \phi^x(y) = \Phi(y-\tilde{x}) = \begin{cases} -\frac{1}{2n} \log |y-\tilde{x}|, & n=2 \\ \frac{1}{n(n-2) \alpha(n) |y-\tilde{x}|^{n-2}}, & n \geq 3 \end{cases}$$

$$= \begin{cases} -\frac{1}{2n} \log |y-x|, & n=2 \\ \frac{1}{n(n-2) \alpha(n) |y-x|^{n-2}}, & n \geq 3 \end{cases} \quad \text{using (4)}$$

$$= \Phi(y-x)$$

$$\therefore \text{if } y \in \partial R_+^n, \text{ then } \phi^x(y) = \Phi(y-\tilde{x}) = \Phi(y-x)$$

$$\left[\begin{array}{l} \text{In the def}^n \text{ of corrector function, } U \subseteq R^n \\ \Delta \phi^x = 0 \quad \text{in } U \\ \Rightarrow -\Delta \phi^x = 0 \quad \text{in } R_+^n \quad [\because R_+^n \subseteq R^n] \end{array} \right]$$

$$\therefore \Delta \phi^x = 0 \quad \text{in } R_+^n$$

$$\left. \begin{array}{l} \phi^x(y) = \Phi(y-\tilde{x}) = \Phi(y-x) \text{ on } \partial R_+^n \end{array} \right\}$$

and $\phi^x(y)$ has no singularity in R_+^n

Thus we get corrector fⁿ ϕ^x build from Φ by reflecting singularity from $x \in R_+^n$ to $\tilde{x} \notin R_+^n$

* Green Function for Half space (i.e. R_+^n) :-

We define Green function for the half space

$$G(x, y) = \Phi(y-x) - \Phi^x(y)$$

$$\therefore G(x, y) = \Phi(y-x) - \Phi(y-\tilde{x})$$

where $x, y \in R_+^n, x \neq y$

and $\tilde{x}$ is reflection of x in ∂R_+^n

Example:- Find the solution of B.V.P

$$\left. \begin{aligned} \Delta u &= 0 & \text{in } R_+^n \\ \text{and } u &= g & \text{on } \partial R_+^n \end{aligned} \right\} \text{---} \textcircled{*}$$

Solⁿ:- Let $u \in C^2(R_+^n)$ be the solⁿ of B.V.P. $\textcircled{*}$

We know that the solⁿ of B.V.P,

$$\Delta u = -f \text{ in } U$$

$$u = g \text{ on } \partial U$$

is given by,

$$u(x) = - \int_{\partial U} g(y) \frac{\partial G(x, y)}{\partial \nu} dS(y) + \int_U f(y) G(x, y) dy$$

where $x \in U$, fixed.

But here in our case, $f=0$ and $U=R_+^n$

$\therefore$ solⁿ of $\textcircled{*}$ is given by

$$u(x) = - \int_{\partial R_+^n} g(y) \frac{\partial G(x, y)}{\partial \nu} dS(y) + 0$$

$$= - \int_{\partial R_+^n} g(y) \frac{\partial G(x, y)}{\partial \nu} dS(y) \text{---} \textcircled{1}$$

where ν is the unit normal drawn outward to the direction R_+^n

Now in half space R_+^n , ν will be in direction of $-y_n$

$$\begin{aligned} \therefore \frac{\partial G(x, y)}{\partial y} &= -\frac{\partial G(x, y)}{\partial y_n} = -\frac{\partial}{\partial y_n} (\Phi(y-x) - \Phi(y-\tilde{x})) \\ &= -\frac{\partial}{\partial y_n} \Phi(y-x) + \frac{\partial \Phi(y-\tilde{x})}{\partial y_n} \quad \text{--- (2)} \end{aligned}$$

$$\text{Now } \Phi(y-x) = \begin{cases} -\frac{1}{2n} \log |y-x| & \text{if } n=2 \\ \frac{1}{n(n-2) \alpha(n) |y-x|^{n-2}} & \text{if } n \geq 3 \end{cases}$$

$$\therefore \text{ for } n=2, \Phi(y-x) = -\frac{1}{2n} \log |y-x|$$

$$\begin{aligned} \frac{\partial \Phi}{\partial y_n}(y-x) &= \frac{\partial}{\partial y_n} \left(-\frac{1}{2n} \log |y-x| \right) \\ &= -\frac{1}{2n} \frac{1}{|y-x|} \frac{\partial |y-x|}{\partial y_n} \\ &= -\frac{1}{2n} \cdot \frac{1}{|y-x|} \cdot \frac{(y_n - x_n)}{|y-x|} \\ &= -\frac{(y_n - x_n)}{2n |y-x|^2} \quad \text{--- (3)} \end{aligned}$$

$$\begin{aligned} \frac{\partial |y-x|}{\partial y_n} &= \frac{\partial}{\partial y_n} \left((y_1 - x_1)^2 + \dots + (y_n - x_n)^2 \right)^{1/2} \\ &= \frac{1}{2} \left((y_1 - x_1)^2 + \dots + (y_n - x_n)^2 \right)^{-1/2} \cdot 2(y_n - x_n) \\ &= \frac{y_n - x_n}{\left((y_1 - x_1)^2 + \dots + (y_n - x_n)^2 \right)^{1/2}} \\ &= \frac{y_n - x_n}{|y-x|} \end{aligned}$$

$$\text{for } n \geq 3, \frac{\partial \Phi}{\partial y_n}(y-x) = \frac{\partial}{\partial y_n} \left(\frac{1}{n(n-2) \alpha(n) |y-x|^{n-2}} \right)$$

$$\begin{aligned} &= \frac{1}{n(n-2) \alpha(n)} \frac{(2-n)}{|y-x|^{n-1}} \cdot \frac{\partial (|y-x|)}{\partial y_n} \\ &= -\frac{1}{n \alpha(n) |y-x|^{n-1}} \cdot \frac{y_n - x_n}{|y-x|} \\ &= -\frac{(y_n - x_n)}{n \alpha(n) |y-x|^n} \quad \text{--- (4)} \end{aligned}$$

combining (3) & (4) we get

$$\therefore \text{ for } n \geq 2, \frac{\partial \Phi(y-x)}{\partial y_n} = \frac{-(y_n - x_n)}{n \alpha(n) |y-x|^n}$$

$$\text{similarly } \frac{\partial \Phi(y-\tilde{x})}{\partial y_n} = \frac{-(y_n + x_n)}{n \alpha(n) |y-\tilde{x}|^n}$$

$$\begin{aligned}
\therefore \frac{\partial G(x,y)}{\partial y_n} &= -\frac{\partial G(x,y)}{\partial y_n} = -\left[\frac{\partial}{\partial y_n} \Phi(y-x) - \frac{\partial}{\partial y_n} \Phi(y-\tilde{x}) \right] \\
&= -\left[\frac{-(y_n-x_n)}{n\alpha(n)|y-x|^n} + \frac{(y_n+x_n)}{n\alpha(n)|y-\tilde{x}|^n} \right] \\
&= \frac{(y_n-x_n)}{n\alpha(n)|y-x|^n} - \frac{(y_n+x_n)}{n\alpha(n)|y-\tilde{x}|^n} \quad \text{--- (5)}
\end{aligned}$$

if $y = (y_1, y_2, \dots, y_n) \in \partial R_+^n$ then $y_n = 0$
and $|y-x| = |y-\tilde{x}|$

$\therefore$ (5) implies

$$\begin{aligned}
\frac{\partial G(x,y)}{\partial y_n} &= \frac{0-x_n - (0+x_n)}{n\alpha(n)|y-x|^n} \\
&= \frac{-2x_n}{n\alpha(n)|y-x|^n}
\end{aligned}$$

Put this value in eqn (4), we have

$$\begin{aligned}
u(x) &= - \int_{\partial R_+^n} g(y) \left(\frac{-2x_n}{n\alpha(n)|y-x|^n} \right) dS(y) \\
\Rightarrow u(x) &= \int_{\partial R_+^n} \frac{2x_n}{n\alpha(n)} \times \frac{g(y)}{|y-x|^n} dS(y), \quad x \in R_+^n \text{ fixed}
\end{aligned}$$

if we define poisson kernel for R_+^n as

$$k(x,y) = \frac{2x_n}{n\alpha(n)|y-x|^n}; \quad x \in R_+^n, y \in \partial R_+^n$$

then

$$u(x) = \int_{\partial R_+^n} g(y) k(x,y) dS(y)$$

This is called Poisson formulae for R_+^n
This is the req soln of B.V.P. (x)

** Poisson's formula for half space:-Statement:- Assume $g \in C(R^{n-1}) \cap L^\infty(R^{n-1})$ and defined u

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial R_+^n} \frac{g(y)}{|x-y|^n} dy, \quad x \in R_+^n$$

Then 1) $u \in C^\infty(R_+^n) \cap L^\infty(R_+^n)$ 2) $\Delta u = 0$ in R_+^n 3) $\lim_{\substack{x \rightarrow x^0 \\ x \in R_+^n}} u(x) = g(x^0)$ for each point $x^0 \in \partial R_+^n$ Proof:- ① For each fixed x the mapping $y \mapsto u(x, y)$ is harmonic, except for $y=x$. As $u(x, y) = u(y, x)$ $\therefore x \mapsto u(x, y)$ is also harmonic, except for $x=y$.Thus $x \mapsto \frac{\partial u}{\partial x_n}(x, y) = k(x, y)$ is harmonic for $x \in R_+^n, y \in \partial R_+^n$

② $k(x, y) = \frac{2x_n}{n\alpha(n)} \cdot \frac{1}{|x-y|^n}$ on integrating

$$\int_{\partial R_+^n} k(x, y) dy = \frac{2x_n}{n\alpha(n)} \int \frac{dy}{|x-y|^n}$$

$$\int_{\partial R_+^n} k(x, y) dy = 1 \quad \text{--- ①}$$

As g is bounded, $u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial R_+^n} \frac{g(y)}{|x-y|^n} dy \quad (x \in R_+^n)$

is also bounded.

since $x \mapsto k(x, y)$ is smooth for $x \neq y$, we easily verify as well $u \in C^\infty(R_+^n)$ with

$$u(x) = \int_{\partial R_+^n} k(x, y) g(y) dy$$

$$\Delta u(x) = \int_{\partial R_+^n} \Delta_x k(x, y) g(y) dy$$

$$\boxed{\Delta u(x) = 0}$$

[$\because k$ is harmonic
 $\Delta_x k = 0$]③ Now fix $x^0 \in \partial R_+^n$, $\epsilon > 0$ choose $\delta > 0$ so small that

that $|g(y) - g(x^0)| < \varepsilon$ if $|y - x^0| < \delta$, $y \in \mathbb{R}_+^n$ (2)
 (By continuity of $g \in C(\mathbb{R}^{n-1})$)

Then if $|x - x^0| < \frac{\delta}{2}$, $x \in \mathbb{R}_+^n$

$$\begin{aligned} |u(x) - g(x^0)| &= \left| \int_{\mathbb{R}_+^n} k(x, y) g(y) dy - g(x^0) \right| = \left| \int_{\mathbb{R}_+^n} k(x, y) g(y) dy - g(x^0) \int_{\mathbb{R}_+^n} k(x, y) dy \right| \\ &= \left| \int_{\mathbb{R}_+^n} k(x, y) (g(y) - g(x^0)) dy \right| \leq \int_{\mathbb{R}_+^n} |k(x, y)| |g(y) - g(x^0)| dy \\ &= \int_{\mathbb{R}_+^n \cap B(x^0, \delta)} |k(x, y)| |g(y) - g(x^0)| dy + \int_{\mathbb{R}_+^n - B(x^0, \delta)} |k(x, y)| |g(y) - g(x^0)| dy \\ &= I + J \quad (\text{say}) \quad \text{--- (3)} \end{aligned}$$

$$\begin{aligned} \text{Now } I &= \int_{\mathbb{R}_+^n \cap B(x^0, \delta)} |k(x, y)| |g(y) - g(x^0)| dy \\ &\leq \varepsilon \int_{\mathbb{R}_+^n} k(x, y) dy \end{aligned} \quad \left[\because y \in B(x^0, \delta) \rightarrow |y - x^0| < \delta \right]$$

$$I \leq \varepsilon \cdot 1 \Rightarrow I \leq \varepsilon \quad \text{--- (4)}$$

Furthermore $|x - x^0| < \frac{\delta}{2}$ & $|y - x^0| \geq \delta$ is $y \notin B(x^0, \delta)$

$$\begin{aligned} \text{then } |y - x^0| &= |y - x + x - x^0| \leq |y - x| + |x - x^0| \\ &\leq |y - x| + \frac{\delta}{2} \\ &\leq |y - x| + \frac{1}{2} |y - x^0| \\ \Rightarrow |y - x| &\geq \frac{1}{2} |y - x^0| \end{aligned}$$

$$\begin{aligned} \text{Now } J &= \int_{\mathbb{R}_+^n - B(x^0, \delta)} |k(x, y)| |g(y) - g(x^0)| dy \\ &\leq 2 \|g\|_{L^\infty} \int_{\mathbb{R}_+^n - B(x^0, \delta)} k(x, y) dy = 2 \|g\|_{L^\infty} \int_{\mathbb{R}_+^n - B(x^0, \delta)} \frac{2^n x_n}{n \alpha(n)} \cdot \frac{1}{|x - y|^n} dy \\ &\leq 2^2 \|g\|_{L^\infty} x_n \int_{\mathbb{R}_+^n - B(x^0, \delta)} \frac{1}{|y - x^0|^n} dy \\ &= \frac{2^{n+2} \|g\|_{L^\infty} x_n}{n \alpha(n)} \int_{\mathbb{R}_+^n - B(x^0, \delta)} |y - x^0|^{-n} dy \end{aligned}$$

$\rightarrow 0$ as $x_n \rightarrow 0^+$
 from eqn (3) we have

$|u(x) - g(x^0)| \leq 2\varepsilon$, provided $|x - x^0|$ is sufficiently small.

$$\Rightarrow \lim_{x \rightarrow x^0} u(x) = g(x^0) \quad \text{for each } x^0 \in \mathbb{R}_+^n \text{ \& } x \in \mathbb{R}_+^n$$

Proved

Note:-

- ① Representation formula for the ball
or Poisson formula for the ball

$$u(x) = \frac{r^2 - |x|^2}{n \alpha(n) r} \int_{\partial B(0, r)} \frac{g(y)}{|y-x|^n} dS(y)$$

- ② Poisson kernel for the ball

$$K(x, y) = \frac{r^2 - |x|^2}{n \alpha(n) r |y-x|^n}$$

ENERGY METHOD

** Uniqueness :- consider first the boundary value problem

$$\left. \begin{array}{l} -\Delta u = f \text{ in } U \\ u = g \text{ on } \partial U \end{array} \right\} \text{ --- } (*)$$

Assume U is open, bounded and ∂U is C^1

Theorem:- (Uniqueness) :- There exist at most one solution

$u \in C^2(\bar{U})$ of

$$\left. \begin{array}{l} -\Delta u = f \text{ in } U \\ u = g \text{ on } \partial U \end{array} \right\} \text{ --- } (1)$$

Proof:- Let u & $\tilde{u}$ both be the solⁿ of given problem (1)

Set $w = u - \tilde{u}$ then $\Delta w = 0$ in U

$$\left[\begin{array}{l} \because \Delta w = \Delta u - \Delta \tilde{u} \\ = f - f \\ = 0 \end{array} \right]$$

By Green formula,

$$\int_U \nabla u \cdot \nabla v \, dx = - \int_U u \Delta v \, dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u \, dS$$

Put $u = w$ $v = w$

$$\int_U \nabla w \cdot \nabla w \, dx = - \int_U w \Delta w \, dx + \int_{\partial U} \frac{\partial w}{\partial \nu} w \, dS$$

but there is no boundary $\left[\because w = u - \tilde{u} = g - g = 0 \text{ on } \partial U \right]$

$$\int_U |\nabla w|^2 \, dx = - \int_U w \Delta w \, dx$$

$$= 0 \quad \left[\because \Delta w = 0 \text{ in } U \right]$$

$$\Rightarrow \quad \nabla w = 0 \quad \text{in } U$$

$$\therefore \quad w = 0 \quad \text{on } \partial U$$

we deduce that $w = u - \tilde{u}$

$$\Rightarrow u - \tilde{u} = 0$$

$$\Rightarrow u = \tilde{u} \quad \text{in } U \quad \underline{\text{proved}}$$

Theorem (Dirichlet's Principle) :-

Assume $u \in C^2(\bar{U})$ solve the B.V. Problem

$$\left. \begin{aligned} -\Delta u &= f & \text{in } U \\ u &= g & \text{on } \partial U \end{aligned} \right\} \quad \text{--- ①}$$

$$\text{Then} \quad I[u] = \min_{w \in A} I[w] \quad \text{--- ②}$$

where $I[w] = \int_U \left(\frac{1}{2} |\nabla w|^2 - w f \right) dx$
and $A = \{w \in C^2(\bar{U}) \mid w = g \text{ on } \partial U\}$

conversely, if $u \in A$ satisfies ② then u solves the boundary value problem ①

or

If $u \in A$ the PDE $-\Delta u = f$ is equivalent to the statement that u minimizes the energy $I[\cdot]$.

Proof:-

We choose $w \in A$, so that $w = g$ on ∂U

$$\left. \begin{aligned} -\Delta u &= f & \text{in } U \\ u &= w & \text{on } \partial U \end{aligned} \right\} \quad \text{--- ③}$$

Then ③ $\Rightarrow$

$$\int_U (-\Delta u - f)(u - w) dx = 0 \quad [\because -\Delta u - f = 0]$$

$$- \int_U \Delta u (u - w) dx - \int_U f(u - w) dx = 0 \quad \text{--- ④}$$

Now applying Green's formula for $(u - w)$ & u .

$$\int_U \nabla u \cdot \nabla v dx = - \int_U u \Delta v dx + \int_{\partial U} \frac{\partial u}{\partial \nu} v ds \quad \left[\begin{array}{l} \text{here } u = u - w \\ \text{ \& } v = u \end{array} \right]$$

$$\int_U \nabla u \cdot \nabla (u - w) dx = - \int_U (u - w) \Delta u dx + \int_{\partial U} \frac{\partial u}{\partial \nu} (u - w) ds$$

$$\int_U \nabla u \cdot \nabla (u - w) dx = - \int_U (u - w) \Delta u dx + 0 \quad \left[\begin{array}{l} u = w = g \\ \therefore u - w = 0 \text{ on } \partial U \end{array} \right]$$

from eqn ③, $\int_U (\nabla u \cdot \nabla(u-w) - f(u-w)) dx = 0$

& there is no boundary term

$$\therefore u-w = g-g = 0 \quad \text{on } \partial U$$

Hence $\int_U (\nabla u \cdot \nabla u - \nabla u \cdot \nabla w - f \cdot u + f \cdot w) dx = 0$

$$\Rightarrow \int_U (|\nabla u|^2 - u f) dx = \int_U (\nabla u \cdot \nabla w - w f) dx$$

$$= \int_U \nabla u \cdot \nabla w dx - \int_U w f dx$$

$$\leq \int_U \frac{1}{2} |\nabla u|^2 dx + \int_U \frac{1}{2} |\nabla w|^2 dx - \int_U w f dx$$

$$\left[\because \nabla u \cdot \nabla w \leq |\nabla u \cdot \nabla w| \leq |\nabla u| |\nabla w| \leq \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |\nabla w|^2 \right]$$

$$\int_U (|\nabla u|^2 - \frac{1}{2} |\nabla u|^2) - u f dx \leq \int_U (\frac{1}{2} |\nabla w|^2 - w f) dx$$

$$\text{or } I[u] \leq I[w] \quad \text{--- ④} \quad [\because w \in A]$$

$\therefore u \in A$ then eqn ③ follows from eqn ④

conversely, let $u \in A$ satisfies, $I[u] = \min_{w \in A} I[w] \quad \text{--- ①}$

Fix any $v \in C_c^\infty(U)$ [v is ∞ differentiable & with compact support]

& write $j(\tau) = I[u + \tau v] \quad \text{--- ②} \quad [\tau \in \mathbb{R}]$

Since $u + \tau v \in A$ for each τ , the scalar function $j(\cdot)$ has a minimum at zero & thus

$$j'(0) = 0 \quad \text{or} \quad \frac{d(j)}{d\tau} = 0$$

[condition for minimize 1st derivative = 0]

$$[I = \frac{d}{d\tau}]$$

Provided this derivative exists. But

$$j(\tau) = I[u + \tau v] = \int_U \left\{ \frac{1}{2} |\nabla u + \tau \nabla v|^2 - (u + \tau v) f \right\} dx$$

$$\quad \quad \quad [\text{By defn of } I[w] = \int_U \frac{1}{2} |\nabla w|^2 - w f dx]$$

$$= \int_U \left\{ \frac{1}{2} |\nabla u|^2 + \tau \nabla u \cdot \nabla v + \frac{\tau^2}{2} |\nabla v|^2 - (u + \tau v) f \right\} dx$$

$$j'(\tau) = 0 + \int_U \nabla u \cdot \nabla v dx + \frac{2\tau}{2} \int_U |\nabla v|^2 dx - 0 - \int_U v f dx$$

$$= \int_U \nabla u \cdot \nabla v dx + \tau \int_U |\nabla v|^2 dx - \int_U v f dx$$

$$j'(0) = \int_U \nabla u \cdot \nabla v dx - \int_U v f dx = \int_U (\nabla u \cdot \nabla v - v f) dx$$

$$\text{but } j'(0) = 0 \Rightarrow \int_U (\nabla u \cdot \nabla v - v f) dx = 0 \quad \text{or} \quad \int_U (-v \Delta u - f v) dx = 0$$

$$\left[\begin{array}{l} \because \text{By Green's formula} \\ \int_U \nabla u \cdot \nabla v \, dx = - \int_U u \Delta v \, dx + \int_{\partial U} u \frac{\partial v}{\partial \nu} \, ds \\ \int_U \nabla u \cdot \nabla v \, dx = - \int_U u \Delta v \, dx \end{array} \right]$$

$$\text{or } \int_U (-\Delta u - f) v \, dx = 0$$

This identity valid for each function $v \in C_c^\infty(U)$ & so,

$$-\Delta u - f = 0 \Rightarrow -\Delta u = f \text{ in } U$$

$$\because u \in A \therefore u = g \text{ on } \partial U$$

hence u satisfies the boundary value problem

$$-\Delta u = f \text{ in } U$$

$$u = g \text{ on } \partial U$$

proved

**** Theorem (smoothness) :-** Let U be an open, bounded subset of $\mathbb{R}^n$. If $u \in C(U)$ and u satisfies the mean value property

$$u(x) = \int_{\partial B(x,r)} u(y) \, ds(y)$$

for every ball $B(x,r) \subset U$, then $u \in C^\infty(U)$

Proof:- Mollifier:- First, we introduce the function η s.t.

$$\eta(x) = \begin{cases} \frac{1}{C e^{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where the constant C is chosen s.t. $\int_{\mathbb{R}^n} \eta(x) \, dx = 1$.

Notice that $\eta \in C^\infty(\mathbb{R}^n)$ and η has compact support.

Now define the function $\eta_\epsilon(x)$ s.t.

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{|x|}{\epsilon}\right)$$

Therefore, $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$ and $\text{supp}(\eta_\epsilon) \subset \{x: |x| < \epsilon\}$

Further $\int_{\mathbb{R}^n} \eta_\epsilon(x) \, dx = 1$

Now choose ϵ s.t. $\epsilon < \text{dist}(x, \partial U)$, Define

$$u_\epsilon(x) = \int_U \eta_\epsilon(x-y) u(y) \, dy$$

Now we claim

$$(1) u_\varepsilon \in C^\infty \quad (2) u_\varepsilon(x) = u(x)$$

First, for (1) $u_\varepsilon \in C^\infty$ because $\eta_\varepsilon \in C^\infty$

We prove (2) as follows. Using the fact that

$$\text{supp } \eta_\varepsilon(x-y) \subset \{y: |x-y| < \varepsilon\}.$$

$$\text{Therefore } u_\varepsilon(x) = \int_{B(x, \varepsilon)} \eta_\varepsilon(x-y) u(y) dy.$$

$$= \frac{1}{\varepsilon^n} \int_{B(x, \varepsilon)} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y) dy.$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \left(\int_{\partial B(x, r)} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y) dS(y) \right) dr$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \left(\int_{\partial B(x, r)} u(y) dS(y) \right) dr$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \left(n \alpha(n) r^{n-1} \int_{\partial B(x, r)} u(y) dS(y) \right) dr$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) n \alpha(n) r^{n-1} u(x) dr$$

$$= u(x) \cdot \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \left(\int_{\partial B(0, r)} dS(y) \right) dr$$

$$= u(x) \cdot \frac{1}{\varepsilon^n} \int_{B(0, \varepsilon)} \eta\left(\frac{|y|}{\varepsilon}\right) dy$$

$$= u(x) \int_{B(0, \varepsilon)} \eta_\varepsilon(y) dy \quad \left[\because \eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{|x|}{\varepsilon}\right) \right]$$

$$= u(x) \cdot 1 \quad \left[\because \int_{\mathbb{R}^n} \eta_\varepsilon(x) dx = 1 \right]$$

$$\boxed{u_\varepsilon(x) = u(x)}$$

Theorem (Harnack's Inequality) :-

For each connected open set

$V \subset\subset U$, there exist a positive

constant C , depending only on V .

s.t.

$$\sup_V u \leq C \inf_V u$$

for all nonnegative harmonic functions u in U .

Thus in particular $\frac{1}{c} u(y) \leq u(x) \leq c u(y)$ for all points $x, y \in V$.

Proof:- Let $\text{dist}(\partial V, V) = 4r$. Let $x \neq y \in V$ be such that $|x - y| \leq r$. Then by the mean value theorem we have

$$\begin{aligned} u(x) &= \int_{B(x, r/2)} u(z) dz = \frac{1}{\alpha(n) r^n 2^n} \int_{B(x, r/2)} u(z) dz \\ &\geq \frac{1}{\alpha(n) r^n 2^n} \int_{B(y, r/4)} u(z) dz \\ &\geq \frac{1}{2^n} \int_{B(y, r/4)} u(z) dz = \frac{u(y)}{2^n} \\ u(x) &\geq \frac{u(y)}{2^n} \end{aligned}$$

Thus $2^n u(y) \geq u(x) \geq \frac{u(y)}{2^n}$ if $x, y \in V$, $|x - y| \leq r$.

There exist a chain of segment of length $\leq r$, of length N (that depends only on $\dim V$ and r), which connects x to y .

$\therefore V$ is connected and $\bar{V}$ is compact, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$ each of which has radius $\frac{r}{2}$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i=2, 3, \dots, N$. Then

$$u(x) \geq \frac{1}{2^{nN}} u(y)$$

for all $x, y \in V$

Proved

Theorem (Estimates on derivatives) :-

Assume u is harmonic in U , then

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0, r))} \quad \text{--- (1)}$$

for each ball $B(x_0, r) \subset U$ and each multiindex α of order $|\alpha| = k$.

$$\begin{aligned} \text{Here } C_0 &= \frac{1}{\alpha(n)} \\ \text{and } C_k &= \frac{(2^{n+1} n k)^k}{\alpha(n)} \end{aligned} \quad \left\{ \begin{array}{l} \text{--- (2)} \\ (k = 1, 2, \dots) \end{array} \right.$$

Proof:- We establish (1) & (2) by induction on k .

Case I:- If $k=0$.

$$\text{Then } |D^\alpha u(x_0)| = |D^k u(x_0)| = |D^0 u(x_0)| = |u(x_0)|$$

By mean value property

$$u(x_0) = \frac{1}{\alpha(n)} \int_{B(x_0, r)} u(y) dy$$

$$|u(x_0)| = \frac{1}{\alpha(n)} \int_{B(x_0, r)} |u(y)| dy$$

$$= \frac{1}{\alpha(n) r^n} \int_{B(x_0, r)} |u| dy$$

$$|u(x_0)| = \frac{C_0}{r^n} \|u\|_{L^1(B(x_0, r))}$$

Case II:- If $k=1$

$$\text{Then } |D^\alpha u(x_0)| = |D u(x_0)| = |u_{x_i}(x_0)|$$

$\because u$ is harmonic

$$\therefore \Delta u = 0$$

$$\frac{\partial}{\partial x_i} \Delta u = 0 \Rightarrow \Delta \left(\frac{\partial u}{\partial x_i} \right) = 0 \Rightarrow \Delta u_{x_i} = 0$$

$\Rightarrow u_{x_i}$ is harmonic.

$$\text{consequently } |u_{x_i}(x_0)| = \left| \frac{1}{\alpha(n)} \int_{B(x_0, r/2)} u_{x_i} dx \right|$$

$$= \left| \frac{1}{\alpha(n) (r/2)^n} \int_{B(x_0, r/2)} u_{x_i} dx \right|$$

$$= \frac{2^n}{\alpha(n) r^n} \int_{\partial B(x_0, r/2)} |u \cdot \nu^i| dS \quad \left[\begin{array}{l} \text{by Gauss} \\ \text{Green's theorem} \end{array} \right] \quad \text{--- (3)}$$

$$|u_{x_j}(x_0)| \leq \frac{2^n}{r} \|u\|_{L^\infty(\partial B(x_0, r_k))}$$

Now if $x \in \partial B(x_0, r_k)$, then $B(x, r_k) \subset B(x_0, r) \subset U$ and

So,

$$|u(x)| \leq \frac{1}{\alpha(n)} \left(\frac{2}{r}\right)^n \|u\|_{L^1(B(x_0, r))}$$

by ① & ② for $k=0$. Combining the inequalities above

We deduce
$$|D^\alpha u(x_0)| \leq \frac{2^{n+1}n}{\alpha(n)} \cdot \frac{1}{r^{n+1}} \|u\|_{L^1(B(x_0, r))}$$

If $|\alpha|=1$. This verifies ①, ② for $k=1$.

② Assume now $k \geq 2$ and ①, ② is valid for balls in U and each multiindex of order less than or equal to $k-1$. Fix $B(x_0, r) \subset U$ and let α be a multiindex with $|\alpha|=k$.

Then $D^\alpha u = (D^\beta u)_{x_j}$ for some $j \in \{1, 2, \dots, n\}$

$|\beta|=k-1$. By calculation similar to those in eqn ③

we establish that

$$|D^\beta u(x)| \leq \frac{nk}{r} \|D^\beta u\|_{L^\infty(\partial B(x_0, r/k))}$$

If $x \in \partial B(x_0, r/k)$, then $B(x, \frac{k-1}{k}r) \subset B(x_0, r) \subset U$

Thus eqn ① & ② for $k-1$ imply

$$|D^\beta u(x)| \leq \frac{(2^{n+1}n(k-1))^{k-1}}{\alpha(n) \left(\frac{k-1}{k}r\right)^{n+k-1}} \|u\|_{L^1(B(x_0, r))}$$

combining the two previous estimates yields the bound

$$|D^\alpha u(x_0)| \leq \frac{(2^{n+1}nk)^k}{\alpha(n) r^{n+k}} \|u\|_{L^1(B(x_0, r))}$$

for $|\alpha|=k$.

□

completed

* Heat Equation:- The partial differential eqn of the form

$$u_t - \Delta u = 0 \quad \text{--- ①}$$

$$\text{and } u_t - \Delta u = f \quad \text{--- ②}$$

are homogeneous and non-homogeneous heat equation respectively.

Here $t > 0$, $x \in U$, where $U \subseteq \mathbb{R}^n$ is open and

u is unknown function defined as

$$u: \bar{U} \times [0, \infty) \rightarrow \mathbb{R} \text{ be such that}$$

$$u = u(x, t)$$

$$\text{where } \Delta u = \Delta_x u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$$

and the function, $f: U \times [0, \infty) \rightarrow \mathbb{R}$ is known function.

* Physical Interpretation of Heat Equation:-

The heat equation which is also known as diffusion equation and u is the density of some quantity such as heat, chemical concentration etc.

if $V \subset U$ be any smooth subregion then the rate of change of the total quantity within V is equal to the negative of net flux through ∂V :-

$$\text{i.e. } \frac{d}{dt} \int_V u dx = - \int_{\partial V} \vec{F} \cdot \vec{\nu} ds. \quad \text{--- ①}$$

$\vec{F}$ being the flux density which is equal to the total quantity or heat going from V .

$$\text{consider eqn ① } \frac{d}{dt} \int_V u dx = - \int_{\partial V} \vec{F} \cdot \vec{\nu} ds$$

$$\Rightarrow \int_V \frac{\partial u}{\partial t} dx = - \int_{\partial V} \vec{F} \cdot \vec{n} dS$$

$$\Rightarrow \int_V \frac{\partial u}{\partial t} dx = - \int_V \operatorname{div} \vec{F} dS \quad \left[\because \text{By Gauss divergence theorem} \right]$$

$$\Rightarrow \int_V \frac{\partial u}{\partial t} dx + \int_V \operatorname{div} \vec{F} dS = 0$$

$$\Rightarrow \int_V \left(\frac{\partial u}{\partial t} + \operatorname{div} \vec{F} \right) dx = 0$$

Since V was arbitrary

$$\therefore \frac{\partial u}{\partial t} + \operatorname{div} \vec{F} = 0$$

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{F} = 0 \quad \text{--- (2)}$$

In many situations $\vec{F}$ is proportional to the gradient of u but point in opposite direction.

[since the flow is from region of higher to lower concentration]

$$\vec{F} \propto -\nabla u$$

$$\Rightarrow \vec{F} = -k \nabla u$$

where $k > 0$, be proportional constant.

$$\Rightarrow \nabla \cdot \vec{F} = \nabla \cdot (-k \nabla u)$$

$$= -k \nabla \cdot (\nabla u) = -k \nabla^2 u$$

$$= -k \Delta u$$

$$\left[\because \nabla = \left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \dots + v_n \frac{\partial}{\partial x_n} \right) \right. \\ \left. \nabla \cdot \nabla = \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right) = \Delta \right]$$

choose k such that $k=1$

$$\therefore \nabla \cdot \vec{F} = -\Delta u$$

using this value in eqn (2), we get

$$\frac{\partial u}{\partial t} - \Delta u = 0$$

$$\text{ie } u_t - \Delta u = 0$$

which is homogeneous heat eqn.

* Derive the fundamental solution of Heat Equation:-

consider the homogeneous heat equation:-

$$u_t - \Delta u = 0 \quad \text{--- (1)}$$

where $u: \bar{U} \times [0, \infty) \rightarrow \mathbb{R}$ is an unknown function

$\therefore u = u(x, t)$, where $x \in \bar{U} \subseteq \mathbb{R}^n$ and U is open and $t > 0$

Let $u(x, t)$ be solⁿ of eqn (1), then $u(\lambda x, \lambda^2 t); \lambda \in \mathbb{R}$ is also a solⁿ of (1).

$$\text{Let } u(x, t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \quad \text{--- (2)} \quad ; \quad x \in \mathbb{R}^n, t > 0$$

where α & β are the constants to be determined and function $v: \mathbb{R}^n \rightarrow \mathbb{R}$ to be determined.

[Now by using (2), we will find partial der. w.r.t 't' and double --- // --- // --- // --- w.r.t x]

$$\begin{aligned} \text{Now } u_t(x, t) &= \frac{\partial}{\partial t} \left(u(x, t) \right) \\ &= \frac{\partial}{\partial t} \left(\frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \right) \\ &= \frac{1}{t^\alpha} \cdot \frac{\partial}{\partial t} \left(v\left(\frac{x}{t^\beta}\right) \right) + v\left(\frac{x}{t^\beta}\right) \cdot \left(-\frac{\alpha}{t^{\alpha+1}} \right) \\ &= \frac{1}{t^\alpha} \cdot \frac{\partial}{\partial t} \left(v\left(\frac{x}{t^\beta}\right) \right) - \frac{\alpha}{t^{\alpha+1}} v\left(\frac{x}{t^\beta}\right) \quad \text{--- (3)} \end{aligned}$$

$$\begin{aligned} \text{also } \frac{\partial}{\partial t} \left(v\left(\frac{x}{t^\beta}\right) \right) &= \frac{\partial}{\partial t} \left(v\left(\frac{x_1}{t^\beta}, \frac{x_2}{t^\beta}, \dots, \frac{x_n}{t^\beta}\right) \right) \\ &= \frac{\partial}{\partial t} (v(y_1, y_2, \dots, y_n)) \quad \text{where } y_i = \frac{x_i}{t^\beta} \quad 1 \leq i \leq n \\ &= \frac{\partial v}{\partial y_1} \frac{\partial y_1}{\partial t} + \frac{\partial v}{\partial y_2} \frac{\partial y_2}{\partial t} + \dots + \frac{\partial v}{\partial y_n} \frac{\partial y_n}{\partial t} \\ &= \frac{\partial v}{\partial y_1} \left(\frac{-\beta x_1}{t^{\beta+1}} \right) + \dots + \frac{\partial v}{\partial y_n} \left(\frac{-\beta x_n}{t^{\beta+1}} \right) \quad \left[\because y_i = \frac{x_i}{t^\beta} \right. \\ &= \frac{-\beta}{t^{\beta+1}} \left(\frac{\partial v}{\partial y_1} x_1 + \dots + \frac{\partial v}{\partial y_n} x_n \right) \quad \left. \frac{\partial y_i}{\partial t} = \frac{-\beta x_i}{t^{\beta+1}} \quad 1 \leq i \leq n \right] \end{aligned}$$

$$= \frac{-\beta}{t^{\beta+1}} \left(\frac{\partial V}{\partial y_1}, \frac{\partial V}{\partial y_2}, \dots, \frac{\partial V}{\partial y_n} \right) (x_1, x_2, \dots, x_n)$$

$$= \frac{-\beta}{t^{\beta+1}} \left(\frac{\partial V}{\partial y_1}, \dots, \frac{\partial V}{\partial y_n} \right) \cdot x$$

$$= \frac{-\beta}{t^{\beta+1}} D_V \left(\frac{x}{t^\beta} \right) \cdot x$$

put this value in eqn (3)

$$u_t(x, t) = \frac{1}{t^\alpha} \left(-\frac{\beta}{t^{\beta+1}} D_V \left(\frac{x}{t^\beta} \right) \cdot x \right) - \frac{\alpha}{t^{\alpha+1}} V \left(\frac{x}{t^\beta} \right)$$

$$= \frac{1}{t^\alpha} \left(-\frac{\beta}{t^{\beta+1}} D_V \left(\frac{x}{t^\beta} \right) \cdot x \right) - \frac{\alpha}{t^{\alpha+1}} V \left(\frac{x}{t^\beta} \right)$$

$$= \frac{-\beta}{t^{\alpha+\beta+1}} \left(D_V \left(\frac{x}{t^\beta} \right) \cdot \frac{x}{t^\beta} \right) - \frac{\alpha}{t^{\alpha+1}} V \left(\frac{x}{t^\beta} \right)$$

$$= \frac{-\beta}{t^{\alpha+1}} (D_V(y) \cdot y) - \frac{\alpha}{t^{\alpha+1}} V(y) \quad \text{--- (4)}$$

$$\text{Now } \Delta u(x, t) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u(x, t)$$

$$= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} u(x, t) \right)$$

$$= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} \left(\frac{1}{t^\alpha} V \left(\frac{x}{t^\beta} \right) \right) \right)$$

$$= \frac{1}{t^\alpha} \sum \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} V \left(\frac{x_1}{t^\beta}, \dots, \frac{x_n}{t^\beta} \right) \right)$$

$$= \frac{1}{t^\alpha} \sum \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} (V(y_1, \dots, y_n)) \right) \quad \text{where } y_i = \frac{x_i}{t^\beta}$$

$$= \frac{1}{t^\alpha} \sum \frac{\partial}{\partial x_i} \left[\frac{\partial V}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_i} + \dots + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial x_i} + \dots + \frac{\partial V}{\partial y_n} \frac{\partial y_n}{\partial x_i} \right]$$

$$= \frac{1}{t^\alpha} \sum \frac{\partial}{\partial x_i} \left[\frac{\partial V}{\partial y_1} \cdot 0 + \dots + \frac{\partial V}{\partial y_i} \left(\frac{1}{t^\beta} \right) + \dots + \frac{\partial V}{\partial y_n} \cdot 0 \right]$$

$$= \frac{1}{t^\alpha} \sum \frac{\partial}{\partial x_i} \left[0 + \dots + \frac{1}{t^\beta} \frac{\partial V}{\partial y_i} + \dots + 0 \right]$$

$$= \frac{1}{t^\alpha} \sum \frac{\partial}{\partial x_i} \left(\frac{1}{t^\beta} \frac{\partial V}{\partial y_i} \right)$$

$$\begin{aligned} \text{if } y_i &= \frac{x_i}{t^\beta} \\ (y_1, y_2, \dots, y_n) &= \left(\frac{x_1}{t^\beta}, \dots, \frac{x_n}{t^\beta} \right) \\ (y_1, \dots, y_n) &= \frac{1}{t^\beta} (x_1, \dots, x_n) \\ \Rightarrow y &= \frac{x}{t^\beta} \end{aligned}$$

⑤

$$= \frac{1}{j^{\alpha+\beta}} \sum \frac{\partial}{\partial x_j} \left(\frac{\partial v}{\partial y_j} \right)$$

$$= \frac{1}{j^{\alpha+\beta}} \sum \frac{\partial}{\partial y_j} \left(\frac{\partial v}{\partial y_j} \right) \cdot \frac{\partial y_j}{\partial x_j} \quad [\text{chain rule}]$$

$$= \frac{1}{j^{\alpha+\beta}} \sum_{j=1}^n \frac{\partial^2 v}{\partial y_j^2} \quad [\because \frac{\partial y_j}{\partial x_j} = \frac{1}{j^\beta}] \quad \text{--- ⑤}$$

put ④ & ⑤ in eqn ①, we get

$$-\frac{\beta}{j^{\alpha+1}} (Dv(y) \cdot y) - \frac{\alpha}{j^{\alpha+1}} v(y) - \frac{1}{j^{\alpha+\beta}} \sum_{j=1}^n \frac{\partial^2}{\partial y_j^2} v\left(\frac{x}{j^\beta}\right) = 0$$

$$-\frac{\beta}{j^{\alpha+1}} (Dv(y) \cdot y) - \frac{\alpha}{j^{\alpha+1}} v(y) - \frac{1}{j^{\alpha+\beta}} \Delta v(y) = 0$$

taking $\beta = \frac{1}{2}$, we get

$$-\frac{1}{2j^{\alpha+1}} Dv(y) \cdot y - \frac{\alpha}{j^{\alpha+1}} v(y) - \frac{1}{j^{\alpha+2 \times \frac{1}{2}}} \Delta v(y) = 0$$

$$-\frac{1}{2j^{\alpha+1}} Dv(y) \cdot y - \frac{\alpha}{j^{\alpha+1}} v(y) - \frac{1}{j^{\alpha+1}} \Delta v(y) = 0$$

$$-\frac{1}{2} Dv(y) \cdot y - \alpha \cdot v(y) - \Delta v(y) = 0 \quad \text{--- ⑥}$$

Now suppose $v(y) = \omega(|y|)$ for some function $\omega: \mathbb{R} \rightarrow \mathbb{R}$
 $= \omega(r)$ where $r = |y| = (\sum y_i^2)^{1/2}$

$$Dv(y) = D(\omega(r)) = \omega'(r) \cdot \frac{\partial}{\partial y} (r)$$

$$= \omega'(r) \cdot \frac{\partial}{\partial y} (|y|)$$

$$= \omega'(r) \cdot \frac{y}{|y|} = \omega'(r) \cdot \frac{y}{r}$$

$$Dv(y) \cdot y = \omega'(r) \frac{y}{r} \cdot y = \omega'(r) \frac{|y|^2}{r} = \omega'(r) \frac{r^2}{r} = r \omega'(r)$$

$$\Delta v(y) = \sum_{j=1}^n \frac{\partial^2}{\partial y_j^2} v(y) = \sum_{j=1}^n \frac{\partial}{\partial y_j^2} (\omega(r))$$

$$= \sum_{j=1}^n \frac{\partial}{\partial y_j} \left(\omega'(r) \cdot \frac{\partial r}{\partial y_j} \right)$$

$$= \sum_{j=1}^n \frac{\partial}{\partial y_j} \left(\omega'(r) \cdot \frac{y_j}{r} \right)$$

$$\left[\begin{array}{l} \because r = |y| = (\sum y_i^2)^{1/2} \\ \frac{\partial r}{\partial y_j} = \frac{1}{2} (\sum y_i^2)^{-1/2} \cdot 2y_j \\ = \frac{y_j}{(\sum y_i^2)^{1/2}} = \frac{y_j}{r} \end{array} \right]$$

$$\begin{aligned}
&= \sum_{j=1}^n \left[\omega'(x_j) \cdot \frac{1}{x_j} + \omega'(x_j) y_j \left(-\frac{1}{x_j^2} \right) \frac{\partial x_j}{\partial y_j} + \frac{y_j}{x_j} \omega''(x_j) \frac{\partial x_j}{\partial y_j} \right] \\
&= \sum_{j=1}^n \left[\frac{\omega'(x_j)}{x_j} + \omega'(x_j) \left(-\frac{y_j}{x_j^2} \right) \cdot \frac{y_j}{x_j} + \frac{y_j}{x_j} \omega''(x_j) \cdot \frac{y_j}{x_j} \right] \\
&= \sum_{j=1}^n \left[\frac{\omega'(x_j)}{x_j} + \omega'(x_j) \left(-\frac{y_j^2}{x_j^3} \right) + \omega''(x_j) \cdot \frac{y_j^2}{x_j^2} \right] \\
&= \sum_{j=1}^n \frac{\omega'(x_j)}{x_j} + \sum_{j=1}^n \omega'(x_j) \left(-\frac{y_j^2}{x_j^3} \right) + \sum_{j=1}^n \omega''(x_j) \frac{y_j^2}{x_j^2} \\
&= \frac{\omega'(x)}{x} \sum_{j=1}^n 1 - \frac{\omega'(x)}{x^3} \sum_{j=1}^n y_j^2 + \frac{\omega''(x)}{x^2} \sum_{j=1}^n y_j^2 \\
&= \frac{\omega'(x)}{x} \cdot n - \frac{\omega'(x)}{x^3} \cdot x^2 + \frac{\omega''(x)}{x^2} \cdot x^2 \\
&= \omega''(x) + \frac{(n-1)}{x} \omega'(x)
\end{aligned}$$

putting these value in eqⁿ (6)

$$-\frac{1}{2} x \omega'(x) - \alpha \omega(x) - \omega''(x) - \frac{(n-1)}{x} \omega'(x) = 0$$

$$\frac{x}{2} \omega'(x) + \alpha \omega(x) + \omega''(x) + \frac{(n-1)}{x} \omega'(x) = 0$$

taking $\alpha = \frac{n}{2}$

$$\frac{x}{2} \omega'(x) + \frac{n}{2} \omega(x) + \omega''(x) + \frac{(n-1)}{x} \omega'(x) = 0$$

multiplying both sides by x^{n+1} , we get

$$\frac{x^n}{2} \omega'(x) + \frac{n x^{n+1}}{2} \omega(x) + x^{n+1} \omega''(x) + \frac{(n-1)}{2} x^{n-2} \omega'(x) = 0$$

$$\frac{1}{2} [x^n \omega(x)]' + [x^{n+1} \omega'(x)]' = 0 \quad \text{--- (7)}$$

integrating both sides

$$\frac{1}{2} x^n \omega(x) + x^{n+1} \omega'(x) = k \quad \text{--- (8)}$$

$$\Rightarrow \frac{\omega(x)}{2} + \frac{\omega'(x)}{x} = \frac{k}{x^{n+1}} \quad \text{--- (9)}$$

Let $x \rightarrow \infty$, then $\frac{\omega(x)}{2} + 0 = 0$

$$\Rightarrow \omega(x) = 0$$

$$\text{eqⁿ 9} \Rightarrow \frac{\omega'(x)}{x} = \frac{k}{x^{n+1}} \Rightarrow \omega'(x) = \frac{k}{x^{n+1}}$$

when $x \rightarrow \infty$, then $w'(x) = \frac{k}{\infty} = 0$

$\therefore$ put $w(x)=0$ & $w'(x)=0$ in eqn (8), we get
 $0+0=k$

$$0=k$$

$\therefore$ eqn (9) implies $\frac{1}{2} w(x) + \frac{w'(x)}{x} = 0$

$$\frac{w'(x)}{x} = -\frac{w(x)}{2}$$

$$\frac{w'(x)}{w(x)} = -\frac{x}{2}$$

Integrating both sides, we get

$$\log w(x) = -\frac{x^2}{4} + \log C$$

$$\Rightarrow \log \frac{w(x)}{C} = -\frac{x^2}{4}$$

$$\Rightarrow \frac{w(x)}{C} = e^{-x^2/4}$$

$$\Rightarrow w(x) = C e^{-x^2/4}$$

$$\text{i.e. } w(|y|) = C e^{-|y|^2/4}$$

$$\Rightarrow v(y) = C e^{-|y|^2/4}$$

$$\Rightarrow v\left(\frac{x}{t^\beta}\right) = C e^{-|x|^2/4t^{2\beta}}$$

$$\Rightarrow u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) = \frac{1}{t^\alpha} (C e^{-|x|^2/4t^{2\beta}})$$

$$\boxed{u(x,t) = \frac{C}{t^{n/2}} e^{-|x|^2/4t}}$$

$$\left[\begin{array}{l} \alpha = n/2 \\ \beta = 1/2 \end{array} \right]$$

This is the solution of heat eqn.

Defⁿ:- The function

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} & (x \in \mathbb{R}^n, t > 0) \\ 0 & (x \in \mathbb{R}^n, t < 0) \end{cases}$$

is called the fundamental solution of heat equation.
 Notice that $\Phi(x,t)$ is singular at point $(0,0)$.
 sometimes we write $\Phi(x,t) = \Phi(|x|,t)$

* Integral of fundamental solution of Heat Equation:- (8)

Lemma:- Prove that for each time $t > 0$

$$\int_{\mathbb{R}^n} \Phi(x, t) dx = 1$$

where $\Phi(x, t)$ is fundamental solⁿ of heat eqⁿ

OR Find the integral of the solution of heat equation

Proof:- Let $I = \int_{\mathbb{R}^n} \Phi(x, t) dx$

from defⁿ of fundamental solution of heat eqⁿ

for $t > 0$, $\Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} \cdot e^{-|x|^2/4t}$

$$\begin{aligned} \therefore I &= \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} \cdot e^{-|x|^2/4t} dx \\ &= \frac{1}{(4\pi t)^{n/2}} \cdot e^{-|x|^2/(2\sqrt{t})^2} dx \quad \text{--- (1)} \end{aligned}$$

put $\frac{x}{2\sqrt{t}} = z \Rightarrow \frac{dx}{2\sqrt{t}} = dz \Rightarrow dx = (2\sqrt{t})^n dz$

$$\begin{aligned} \therefore \text{eqn (1)} \Rightarrow I &= \frac{1}{(4\pi t)^{n/2}} \cdot \int_{\mathbb{R}^n} e^{-|z|^2} (2\sqrt{t})^n dz \\ &= \frac{(2\sqrt{t})^n}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} dz = \frac{2^n t^{n/2}}{2^n \pi^{n/2} t^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} dz \end{aligned}$$

$$= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} dz$$

$$= \frac{1}{\pi^{n/2}} \prod_{j=1}^n \int_{\mathbb{R}^n} e^{-z_j^2} dz_j$$

$$= \frac{1}{\pi^{n/2}} \prod_{j=1}^n \int_{-\infty}^{\infty} e^{-z_j^2} dz_j$$

~~Now consider~~ $= \frac{1}{\pi^{n/2}} \prod_{j=1}^n \int_0^{\infty} e^{-z_j^2} dz_j$

Now consider $\int_0^{\infty} e^{-z_j^2} dz_j$

put $z_j^2 = t \Rightarrow 2z_j dz_j = dt \Rightarrow dz_j = \frac{dt}{2z_j} = \frac{dt}{2\sqrt{t}}$

$z=0 \Rightarrow t=0$ and $z \rightarrow \infty \Rightarrow t \rightarrow \infty$

$$\begin{aligned} \therefore \int_0^\infty e^{-z_i^2} dz_i &= \int_0^\infty e^{-t} \frac{dt}{\sqrt{t}} = \frac{1}{\sqrt{t}} \int_0^\infty t^{-1/2} e^{-t} dt \\ &= \frac{1}{\sqrt{t}} \Gamma(-\frac{1}{2}+1) \\ &= \frac{1}{\sqrt{t}} \Gamma(\frac{1}{2}) = \frac{1}{\sqrt{t}} \sqrt{\pi} \end{aligned} \quad \text{③}$$

[$\because \int_0^\infty x^{n-1} e^{-x} dx = \Gamma(n)$
and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$]

using this value in ②, we get

$$I = \frac{1}{\pi^{n/2}} \prod_{i=1}^n \left(\frac{\sqrt{\pi}}{2} \right) = \frac{1}{\pi^{n/2}} \prod_{i=1}^n \sqrt{\pi}$$

$$= \frac{1}{\pi^{n/2}} (\sqrt{\pi} \times \sqrt{\pi} \times \dots \text{ n times})$$

$$= \frac{1}{\pi^{n/2}} (\sqrt{\pi})^n = \frac{1}{\pi^{n/2}} \cdot \pi^{n/2} = 1$$

Hence

$$\boxed{\int_{\mathbb{R}^n} \Phi(x, t) dx = 1}$$

* Initial value Problem:-

consider the initial value (or Cauchy) Problem.

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}$$

Suppose that the function $(x, t) \mapsto \Phi(x, t)$ solves the heat equation away from the singularity at $(0, 0)$.

then $(x, t) \mapsto \Phi(x-y, t)$, for fixed $y \in \mathbb{R}^n$

will also solve the heat eqⁿ away from singularity.

consequently the convolution

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}^n} \Phi(x-y, t) g(y) dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y) dy \quad (x \in \mathbb{R}^n, t > 0) \end{aligned}$$

should also be a solution.

* Theorem (Solution of Initial value Problem):-

Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and u is given by

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x-y, t) g(y) dy \quad \text{--- (i)}$$

Then ① $u \in C^\infty(\mathbb{R}^n \times (0, \infty))$

$$\text{② } u_t - \Delta u = 0 \quad (x \in \mathbb{R}^n, t > 0)$$

$$\text{③ } \lim_{\substack{(x, t) \rightarrow (x^0, 0) \\ x \in \mathbb{R}^n, t > 0}} u(x, t) = g(x^0) \text{ for each } x^0 \in \mathbb{R}^n.$$

Proof:- (i) since the function $\frac{1}{t^{n/2}} e^{-|x|^2/4t}$ is infinitely differentiable and has uniformly bounded derivatives of all orders on $\mathbb{R}^n \times [\delta, \infty)$ for each $\delta > 0$, so $u \in C^\infty(\mathbb{R}^n \times (0, \infty))$

(ii) According to Leibniz taking partial der. w.r.t 't' and $x \in \mathbb{R}^n$ --- (ii)

$$u_t(x, t) = \int_{\mathbb{R}^n} \Phi_t(x-y, t) g(y) dy$$

$$\text{and } \Delta u(x, t) = \int_{\mathbb{R}^n} \Delta_x \Phi(x-y, t) g(y) dy$$

$$\Rightarrow u_t(x, t) - \Delta u(x, t) = \int_{\mathbb{R}^n} (\Phi_t(x-y, t) - \Delta_x \Phi(x-y, t)) g(y) dy \quad \text{--- (iii)}$$

since Φ is fundamental solⁿ of heat eqⁿ
 $\therefore \Phi_t - \Delta_x \Phi = 0$

$$\therefore \text{eqⁿ (iii)} \Rightarrow u_t(x, t) - \Delta u(x, t) = \int_{\mathbb{R}^n} 0 \cdot g(y) dy$$

$$\boxed{u_t(x, t) - \Delta u(x, t) = 0} \text{ where } x \in \mathbb{R}^n \text{ \& } t > 0$$

③ fix $x^0 \in \mathbb{R}^n$ and since $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

$\therefore g$ is continuous in $\mathbb{R}^n$.

$\Rightarrow g$ is continuous at x^0 ($\because x^0 \in \mathbb{R}^n$)

$\Rightarrow$ for given $\epsilon > 0$, $\exists$ a $\delta > 0$ such that

$$|g(y) - g(x^0)| < \epsilon$$

(iv) whenever $|y - x^0| < \delta$ where $y \in \mathbb{R}^n$

Thus if $|x - x^0| < \delta/2$ --- (*)

then we have

$$\begin{aligned}
 |u(x,t) - g(x^0)| &= \left| \int_{\mathbb{R}^n} \Phi(x-y, t) g(y) dy - g(x^0) \right| \\
 &= \left| \int_{\mathbb{R}^n} \Phi(x-y, t) g(y) dy - \int_{\mathbb{R}^n} \Phi(x-y, t) g(x^0) dy \right| \\
 &\quad \left[\because \int_{\mathbb{R}^n} \Phi(x, t) dx = 1 \right. \\
 &\quad \left. \Rightarrow \int_{\mathbb{R}^n} \Phi(x-y, t) g(x^0) dy = g(x^0) \right] \\
 &= \left| \int_{\mathbb{R}^n} (g(y) - g(x^0)) \Phi(x-y, t) dy \right| \\
 &\leq \int_{\mathbb{R}^n - B(x^0, \delta)} |g(y) - g(x^0)| \Phi(x-y, t) dy \\
 &\quad + \int_{B(x^0, \delta)} |g(y) - g(x^0)| \Phi(x-y, t) dy \\
 &= \text{I} + \text{II}
 \end{aligned}$$

$$\text{Now } \text{II} = \int_{B(x^0, \delta)} |g(y) - g(x^0)| \Phi(x-y, t) dy \leq \epsilon \int_{\mathbb{R}^n} \Phi(x-y, t) dy = \epsilon \cdot 1$$

$$\text{I} \leq \epsilon \quad \text{--- (V)}$$

Now if $y \in \mathbb{R}^n - B(x^0, \delta)$ then $y \notin B(x^0, \delta)$

$$\Rightarrow |y - x^0| > \delta \Rightarrow \frac{\delta}{2} < \frac{|y - x^0|}{2}$$

$$\begin{aligned} \text{Now } |y - x^0| &= |y - x + x - x^0| \leq |y - x| + |x - x^0| \\ &< |y - x| + \delta/2 \\ &< |y - x| + \frac{|y - x^0|}{2} \end{aligned}$$

[By eqn (4)]

$$\Rightarrow \frac{|y - x^0|}{2} < |y - x|$$

$$\text{consequently } \text{I} = \int_{\mathbb{R}^n - B(x^0, \delta)} \Phi(x-y, t) |g(y) - g(x^0)| dy$$

$$\begin{aligned}
 &\leq 2 \|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n - B(x^0, \delta)} \Phi(x-y, t) dy \\
 &\leq 2 \|g\|_{L^\infty(\mathbb{R}^n)} \cdot \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n - B(x^0, \delta)} \exp\left(-\frac{|x-y|^2}{4t}\right) dy
 \end{aligned}$$

$$\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(x^0, \delta)} \exp\left(-\frac{|x-y|^2}{4t}\right) dy \quad \text{where } C = \frac{2 \|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi)^{n/2}}$$

$$\begin{aligned}
 &\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(x^0, \delta)} \exp\left(-\frac{|y - x^0|^2}{16t}\right) dy \\
 &\quad \left[\because \frac{|y - x^0|}{2} < |y - x| \right. \\
 &\quad \Rightarrow \frac{|y - x^0|^2}{4} < |y - x|^2 \\
 &\quad \Rightarrow \frac{1}{4} - \frac{1}{16t} |y - x^0|^2 > -\frac{1}{4t} |x - y|^2
 \end{aligned}$$

$$= \frac{C}{t^{n/2}} \int_{\delta}^{\infty} \exp\left(-\frac{y^2}{4t}\right) \cdot y^{n-1} dy$$

$$\left[\begin{array}{l} |y-x^0| = y \\ \text{where } y \text{ value} \\ \text{from } \delta \text{ to } \infty \end{array} \right]$$

$$\rightarrow 0 \text{ as } t \rightarrow 0$$

$$x \in J \rightarrow 0 \text{ at } t \rightarrow 0$$

Thus for any $\varepsilon > 0$, $\exists$ some δ s.t if $|x-x^0| < \frac{\delta}{2}$ and t is small enough, then

$$|u(x,t) - g(x^0)| < \varepsilon$$

This Proves $\lim_{(x,t) \rightarrow (x^0,0)} u(x,t) = g(x^0)$ for each point $x^0 \in \mathbb{R}^n$
 $x \in \mathbb{R}^n, t > 0$

** Non-Homogeneous Initial value Problem :-

The non-homogeneous initial-value problem

$$u_t - \Delta u = f \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

$$u = 0 \quad \text{on } \mathbb{R}^n \times \{t=0\}$$

$$\text{i.e. } u(x,0) = 0 \quad \forall x \in \mathbb{R}^n$$

$$\text{where } f \in C_1^2(\mathbb{R}^n \times [0, \infty))$$

and f has a compact support.

* Solution of Non-homogeneous Problem:-

Let $f \in C_1^2(\mathbb{R}^n \times [0, \infty))$ and f has a compact support

Define
$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) dy ds$$

$$= \int_0^t \frac{1}{4\pi(t-s)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) f(y,s) dy ds$$

then prove that

$$\text{where } x \in \mathbb{R}^n, t > 0$$

$$(i) \quad u \in C_1^2(\mathbb{R}^n \times (0, \infty))$$

$$(ii) \quad u_t(x,t) - \Delta u(x,t) = f(x,t) \quad ; \quad x \in \mathbb{R}^n, t > 0$$

$$(iii) \quad \lim_{(x,t) \rightarrow (x^0,0)} u(x,t) = 0 \quad \text{for each point } x^0 \in \mathbb{R}^n$$

Proof:- (i) By changing the variable, we write

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s) dy ds \quad \text{--- (1)}$$

since $f \in C_c^2(\mathbb{R}^n \times (0, \infty))$ has compact support and $\Phi(y,s)$ is smooth near $s=t > 0$, we compute by Leibniz rule

$$u_t(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y, t-s) dy ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y, 0) dy$$

$$\text{and } \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i \partial x_j} f(x-y, t-s) dy ds \Phi(y,s) \quad (i, j = 1, \dots, n)$$

Thus $u_t, \partial_{x_i}^2 u$ and $u, \partial_{x_i} u$ belong to $C(\mathbb{R}^n \times (0, \infty))$.

(ii) We calculate

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y, t-s) dy ds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y, 0) dy - \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \Delta_x f(x-y, t-s) dy ds \\ &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \left[\frac{\partial}{\partial t} - \Delta_x \right] f(x-y, t-s) dy ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y, 0) dy \\ &= \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \left[-\frac{\partial}{\partial s} - \Delta_y \right] f(x-y, t-s) dy ds \\ &\quad + \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s) \left[-\frac{\partial}{\partial s} - \Delta_y \right] f(x-y, t-s) dy ds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y, 0) dy \\ &= I_\epsilon + J_\epsilon + K \quad \text{--- (2)} \end{aligned}$$

By Integral of fundamental solution

$$\begin{aligned} |J_\epsilon| &= \left| \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y \right) f(x-y, t-s) \right] dy ds \right| \\ &\leq (\|f_t\|_{L^\infty} + \|\Delta^2 f\|_{L^\infty}) \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s) dy ds = 1 \end{aligned}$$

$$\leq C \int_0^\varepsilon 1 dy$$

$$\left[\because \int_{\mathbb{R}^n} \Phi(y, s) dy = 1 \right]$$

(14)

$$\leq \varepsilon C$$

$$|J_\varepsilon| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

$$|I_\varepsilon| = \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y, s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y \right) f(x-y, t-s) \right] dy ds$$

$$- \int_{\mathbb{R}^n} \left[\frac{d}{ds} \left(-\frac{\partial}{\partial s} f(x-y, t-s) \right) \int \Phi(y, s) ds \right] dy$$

$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y, s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y \right) f(x-y, t-s) \right] dy ds$$

$$- \int_{\mathbb{R}^n} \left[\Phi(y, s) f(x-y, t-s) dy \right]_\varepsilon^t$$

$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y, s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y \right) f(x-y, t-s) \right] dy ds -$$

$$\int_{\mathbb{R}^n} \Phi(y, t) f(x-y, 0) dy + \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy$$

$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y, s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y \right) f(x-y, t-s) \right] dy ds - k +$$

$$\int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy$$

$$J_\varepsilon = \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy - k$$

$$J_\varepsilon + k = \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy$$

from eqn (2) $u_t(x, t) - \Delta u(x, t) = \varepsilon C + \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy$

$$= \lim_{\varepsilon \rightarrow 0} \left[\varepsilon C + \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy \right]$$

$$= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy$$

$$= \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t) dy$$

$$= f(x, t)$$

$$\lim_{\varepsilon \rightarrow 0} (u_t(x, t) - \Delta u(x, t)) = f(x, t) \quad f$$

$$\|u(\cdot, t)\|_{L^\infty} \leq \|f\|_{L^\infty} \rightarrow 0 \text{ as } t \rightarrow 0$$

* Solution of Homogeneous problem with general initial data: (13)

The Homogeneous problem with general initial data is:-

$$u_t - \Delta u = f \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}$$

The solⁿ of this problem will be given by combining the both previous theorem and it's given by:-

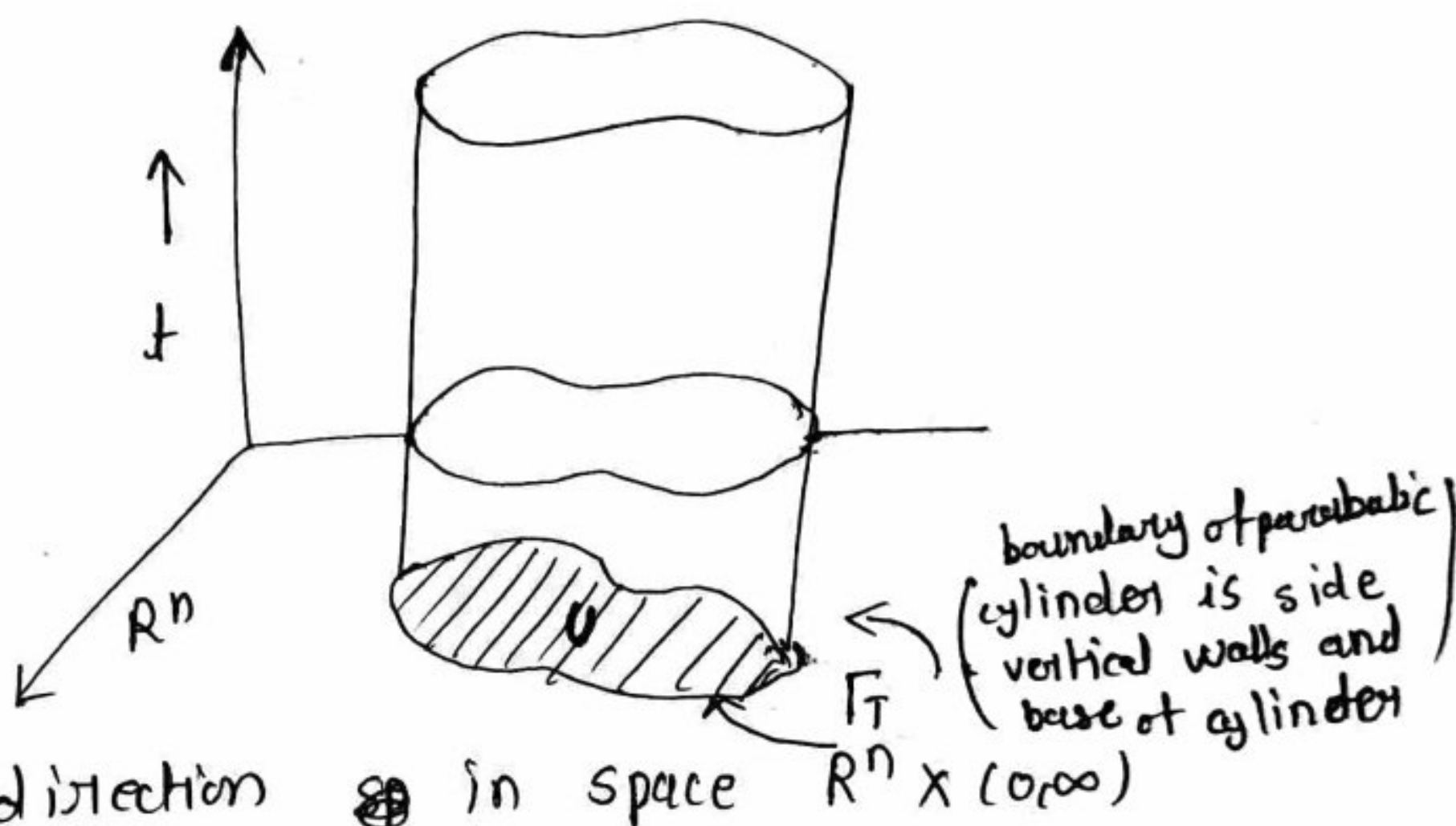
$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x-y, t) f(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y, s) dy ds$$

Notations:-

Assume $U \subset \mathbb{R}^n$ is open and bounded and fix a time $T > 0$

Here to show region U_T we thought of a fig of parabolic cylinder, where

t is taken in upward direction in space $\mathbb{R}^n \times (0, \infty)$



Here

$\Gamma_T \rightarrow$ parabolic boundary of U_T

(which are the vertical walls and base of cylinder)

and $U_T \rightarrow$ is interior and top of cylinder

Definition: Parabolic cylinder:-

We define the parabolic cylinder as

$$U_T := U \times [0, T]$$

Here U_T as being the parabolic interior of $\bar{U} \times [0, T]$ and careful note that U_T includes the top $U \times \{t=T\}$

Definition: Parabolic Boundary:-

The parabolic boundary of U_T is defined by

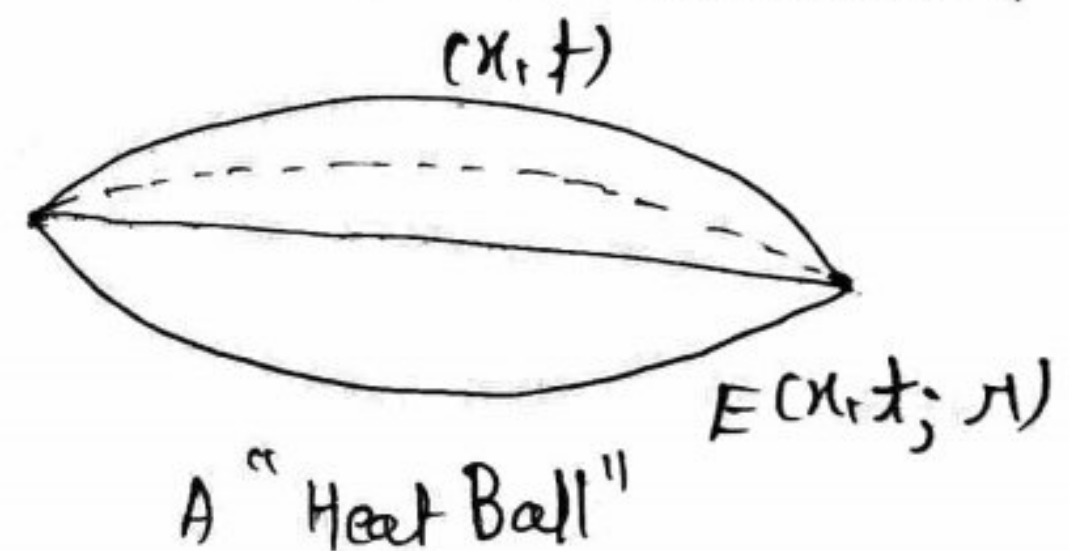
$$\Gamma_T = \overline{U_T} - U_T$$

The parabolic boundary Γ_T comprises $\left[\begin{array}{l} \text{Here} \\ \overline{U_T} = \overline{U} \times [0, T] \end{array} \right]$ the bottom and vertical sides of $U \times [0, T]$ but not the top.

Definition: Heat Ball:- For fixed $x \in \mathbb{R}^n, t \in \mathbb{R}, \tau > 0$ we define

$$E(x, t; \tau) = \left\{ (y, s) \in \mathbb{R}^{n+1}; s \leq t, \Phi(x-y, t-s) \geq \frac{1}{\tau^n} \right\}$$

Then $E(x, t; \tau)$ is called a "Heat ball". Here Φ is fundamental soln of Heat eqn.



This is a region in space-time, the boundary of which is a level set of $\Phi(x-y, t-s)$, Note that point (x, t) is at the centre of the top.

Theorem:- Mean value formula for Heat Equation:-

Let $u \in C^2(U_T)$ is a solution of Heat equation, Then

$$u(x, t) = \frac{1}{4\tau^n} \iint_{E(x, t; \tau)} u(y, s) \frac{|x-y|^2}{(t-s)^2} dy ds$$

for each ball $E(x, t; \tau) \subset U_T$

$$C_1^2(U) = \{ u: U_T \rightarrow \mathbb{R} \mid u, \partial_x u, \partial_x^2 u, u_t \in C(U) \}$$

Proof:- consider the heat ball

$$E(x, t; \tau) = \left\{ (y, s) \in \mathbb{R}^{n+1} \mid s \leq t, \Phi(x-y, t-s) \geq \frac{1}{\tau^n} \right\}$$

where Φ is the fundamental solution of heat equation (17) defined by

$$\Phi(x, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & ; x \in \mathbb{R}^n, t > 0 \\ 0 & ; x \in \mathbb{R}^n, t < 0 \end{cases}$$

Now we shift the space and time coordinates so that $x=0, t=0$ and define

$$E(\tau) = E(0, 0, \tau)$$

$$\text{and let } \phi(\tau) = \frac{1}{\tau^n} \iint_{E(\tau)} u(y, s) \frac{|y|^2}{s^2} dy ds \quad \text{--- (1)}$$

We prove that $\phi(\tau) = H u(0, 0)$.

put $y = \tau y'$ and $s = \tau^2 s'$

then $E(\tau) \rightarrow E(1)$

(i.e. Heat ball change in $E(1)$)

$$\Rightarrow dy = \tau^n dy' \text{ and } ds = \tau^2 ds'$$

$$\text{and } dy ds = \tau^{n+2} dy' ds'$$

$$\text{and } \frac{|y|^2}{s^2} = \frac{\tau^2 |y'|^2}{\tau^4 s'^2} = \frac{1}{\tau^2} \frac{|y'|^2}{s'^2}$$

$\therefore$ using these values in (1), we get

$$\phi(\tau) = \frac{1}{\tau^n} \iint_{E(1)} u(\tau y', \tau^2 s') \frac{1}{\tau^2} \frac{|y'|^2}{(s')^2} \tau^{n+2} dy' ds'$$

$$= \iint_{E(1)} u(\tau y', \tau^2 s') \frac{|y'|^2}{(s')^2} dy' ds'$$

differentiating w.r.t τ both sides, we get

$$\phi'(\tau) = \iint_{E(1)} \frac{\partial}{\partial \tau} (u(\tau y', \tau^2 s')) \frac{|y'|^2}{(s')^2} dy' ds' \quad \text{--- (2)}$$

$$= \iint_{E(1)} \left(\sum_{i=1}^n u_{y_i} y_i' + 2\tau s' u_s \right) \cdot \frac{|y'|^2}{(s')^2} dy' ds'$$

again changing variables y', s' to y and s

$$\phi'(\tau) = \iint_{E(\tau)} \left[\sum_{i=1}^n u_{y_i} \frac{y_i}{\tau} + \frac{2s}{\tau} u_s \right] \frac{|y|^2}{s^2 \tau^n} dy ds \quad [\text{using } (*) (**)]$$

$$= \frac{1}{\tau^{n+1}} \iint_{E(\tau)} \sum_{i=1}^n u_{y_i} y_i \frac{|y|^2}{s^2} dy ds + \frac{1}{\tau^{n+1}} \iint_{E(\tau)} 2|y|^2 \frac{u_s}{s} dy ds$$

$$\phi'(r) = A+B$$

(8)

Now define $\psi = -\frac{n}{2} \log(-4rs) + \frac{|y|^2}{4s} + n \log r$

and $\psi = 0$ on $\partial E(r)$

$$B = \frac{1}{r^{n+1}} \iint_{E(r)} 2u_s \frac{|y|^2}{s} dy ds$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \sum_{i=1}^n y_i \psi_{y_i} dy ds$$

$$\begin{aligned} \psi_{y_i} &= \frac{1}{4s} \cdot \frac{\partial}{\partial y_i} |y|^2 \\ &= \frac{1}{4s} \frac{\partial}{\partial y_i} [y_1^2 + y_2^2 + \dots + y_n^2] \\ &= \frac{1}{4s} \times 2y_i = \frac{y_i}{2s} \\ \sum_{i=1}^n y_i \psi_{y_i} &= \frac{|y|^2}{2s} \end{aligned}$$

Integration by parts formula

$$\int_U u x_i v dx = - \int_U u v x_i dx + \int_{\partial U} u v x_i^j ds$$

$$B = \frac{1}{r^{n+1}} \iint_{E(r)} 4 \sum_{i=1}^n y_i \psi_{y_i} u_s dy ds$$

$$= -\frac{1}{r^{n+1}} \iint_{E(r)} 4 \sum_{i=1}^n \left\{ u_{s y_i} y_i \psi + u_s \cdot 1 \cdot \psi \right\} dy ds + \iint_{\partial E(r)} u_s y_i \psi dy ds$$

$$= -\frac{1}{r^{n+1}} \iint_{E(r)} \left(4 \sum_{i=1}^n u_{s y_i} y_i \psi + 4 \sum_{i=1}^n u_s \psi \right) dy ds + 0 \quad [\because \psi = 0 \text{ on } \partial E(r)]$$

$$= -\frac{1}{r^{n+1}} \iint_{E(r)} 4n u_s \psi dy ds - \frac{1}{r^{n+1}} \iint_{E(r)} 4 \sum_{i=1}^n u_{s y_i} y_i \psi dy ds$$

$$= -\frac{1}{r^{n+1}} \iint_{E(r)} 4n u_s \psi dy ds + \frac{1}{r^{n+1}} \iint_{E(r)} 4 \sum_{i=1}^n y_i u_{y_i} \psi_s dy ds$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} \left[-4n u_s \psi + 4 \sum_{i=1}^n u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2} \right) \right] dy ds$$

using Integration by parts

$$\begin{aligned} \iint_{E(r)} u_{s y_i} \psi dy ds &= - \iint_{E(r)} u_{y_i} \psi_s dy ds \\ &\quad + \iint_{\partial E(r)} u_{y_i} \psi ds dy \\ &= - \iint_{E(r)} u_{y_i} \psi_s dy ds \end{aligned}$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} \left[-4n u_s \psi - 4 \sum_{i=1}^n u_{y_i} y_i \frac{n}{2s} - 4 \sum_{i=1}^n u_{y_i} y_i \frac{|y|^2}{4s^2} \right] dy ds$$

$$B = \frac{1}{r^{n+1}} \iint_{E(r)} \left(-4n u_s \psi - 4 \sum_{i=1}^n u_{y_i} y_i \frac{n}{2s} \right) dy ds - A$$

from eqn (8): $\phi'(r) = \frac{1}{r^{n+1}} \iint_{E(r)} \left(-4n u_s \psi - 4 \sum_{i=1}^n u_{y_i} y_i \frac{n}{2s} \right) dy ds + A - A$

$$\phi'(x) = \frac{1}{x^{n+1}} \iint_{E(x)} \left(-4n u_s \psi - \frac{2n}{s} \sum_{i=1}^n u_{y_i} y_i \right) dy ds$$

$\therefore u$ solves the heat equation

$$= \frac{1}{x^{n+1}} \iint_{E(x)} \left(-4n \Delta_y u \psi - \frac{2n}{s} \sum_{i=1}^n u_{y_i} y_i \right) dy ds$$

$$\left[\begin{array}{l} \because u_t = \Delta_x u \\ u_s = \Delta_y u \end{array} \right]$$

[By integration by parts integrating w.r.t y

$$\iint_{E(x)} \Delta_y u \psi dy ds = - \iint_{E(x)} u_{y_i} \psi_{y_i} dy ds + \iint_{\partial E(x)} u \cdot \psi dy ds]$$

$$\phi'(x) = \frac{1}{x^{n+1}} \iint_{E(x)} \left(4n \sum_{i=1}^n u_{y_i} \psi_{y_i} - \frac{2n}{s} \sum_{i=1}^n u_{y_i} y_i \right) dy ds$$

$$= \sum_{i=1}^n \frac{1}{x^{n+1}} \iint_{E(x)} \left(4n u_{y_i} \psi_{y_i} - \frac{2n}{s} u_{y_i} y_i \right) dy ds$$

$$\phi'(x) = \sum_{i=1}^n \frac{1}{x^{n+1}} \iint_{E(x)} \left(4n u_{y_i} \frac{x y_i}{2s} - \frac{2n}{s} u_{y_i} y_i \right) dy ds$$

$$\boxed{\phi'(x) = 0}$$

This means that ϕ is constant, then

$$\phi(x) = \lim_{t \rightarrow 0} \phi(t) = u(0,0) \lim_{t \rightarrow 0} \frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dy ds$$

$$\boxed{\phi(x) = 4 u(0,0)}$$

$$\left[\begin{array}{l} \because \frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dy ds \\ \iint_{E(1)} \frac{|y|^2}{s^2} dy ds = 4 \end{array} \right]$$

complete

Theorem:- (Maximum Principle for the heat equation)

(23)

Assume $u \in C^2(U_T) \cap C(\bar{U}_T)$ solves heat equation in U_T .

① Then $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$

$$\Gamma_T = \bar{U}_T - U_T = \text{Parabolic boundary of } U_T$$

② Furthermore if U is connected

and there exist a point $(x_0, t_0) \in U_T$

such that $u(x_0, t_0) = \max_{\bar{U}_T} u$

then u is constant in $\bar{U}_{t_0}$.

Proof:- Suppose there exists $(x_0, t_0) \in U_T$, with $u(x_0, t_0) = \max_{\bar{U}_T} u = M$

Pick π small enough so that $E(x_0, t_0, \pi) \subset U_T$

Applying mean value property,

$$M = u(x_0, t_0) = \frac{1}{4\pi^n} \iint_{E(x_0, t_0, \pi)} u(y, s) \frac{|x_0 - y|^2}{(t_0 - s)^2} ds ds \leq M. \quad \text{--- ①}$$

Then equality holds only if

$$u(y, s) = M \text{ for all } (y, s) \in E(x_0, t_0, \pi) \quad \text{--- ②}$$

Now we have proved u is constant within in a small heat ball. Then we try to extend the result to the whole space.

First we draw any segment L in U_T connecting (x_0, t_0) with some other $(y_0, s_0) \in U_T$, with $s_0 < t_0$ and we want to prove $u \equiv M$ on L . Consider

$$\tau_0 = \min \{ s \geq s_0 \mid u(x, t) = M \text{ \# points } (x, t) \in L, s \leq t \leq t_0 \}$$

Assume $\tau_0 > s_0$. Then $u(z_0, \tau_0) = M$ for some point (z_0, τ_0) on $L \cap U_T$ and so $u = M$ for all sufficiently small $\pi > 0$. It contradicts that $E(z_0, \tau_0, \pi)$ contains $L \cap \{\tau_0 - \sigma \leq t \leq \tau_0\}$ for some $\sigma > 0$.

Thus $\tau_0 = s_0$ and $u = M$ on L

Next, we fix a point $x \in U$ and any time $0 \leq t \leq t_0$.
 We can connect the point to (x_0, t_0) with a finite number of line segments in U_T . According to the first step $u = M$ on each segment so $u(x, t) = M$. Then u is constant over the whole space.

Uniqueness for
 an Cauchy problem

* Theorem (Maximum Principle for the initial value Problem) :-

Suppose $u \in C^2(\mathbb{R}^n \times [0, T]) \cap C(\mathbb{R}^n \times [0, T])$ solves

$$\begin{aligned} u_t - \Delta u &= f & \text{in } \mathbb{R}^n \times (0, T) \\ u &= g & \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}$$

and satisfies the growth estimate
 $u(x, t) \leq A e^{a|x|^2}$ ($x \in \mathbb{R}^n, 0 \leq t \leq T$)

for constant $A, a > 0$. Then

$$\sup_{\mathbb{R}^n \times [0, T]} u = \sup_{\mathbb{R}^n} g$$

Proof:-

To make the proof simpler, we impose specific restriction on time T .

First assume $4aT < 1$ — (1)

fix $y \in \mathbb{R}^n, u > 0$ and define

$$v(x, t) = u(x, t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} \exp\left(\frac{|x-y|^2}{4(T+\varepsilon-t)}\right) \quad \text{--- (2)}$$

Then calculation show that $v(x, t)$ satisfies

$$v_t - \Delta v = 0 \text{ in } \mathbb{R}^n \times [0, T] \quad \text{--- (3)}$$

Fix $\mu > 0$ and set $U = B^0(y, \mu), U_T = B^0(y, \mu) \times [0, T]$

According to the maximum principle

$$\max_{\overline{U_T}} u = \max_{\Gamma_T} u \quad \text{--- (4)}$$

Now if $x \in \mathbb{R}^n$

$$v(x, 0) = u(x, 0) - \frac{\mu}{(T+\varepsilon)^{n/2}} \exp\left(\frac{|x-y|^2}{4(T+\varepsilon)}\right)$$

$$\leq u(x, 0) = g(x)$$

(22)

If $|x-y| = r$, $0 \leq t \leq T$ then

$$v(x, t) = u(x, t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} \exp\left(\frac{|x-y|^2}{4(T+\varepsilon-t)}\right)$$

$$\leq A e^{\alpha|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2}} \exp\left(\frac{|x-y|^2}{4(T+\varepsilon-t)}\right)$$

$$\leq A \exp(\alpha(|y|+r)^2) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} \exp\left(\frac{|x-y|^2}{4(T+\varepsilon-t)}\right)$$

Since there exist $\delta > 0$ s.t. $\frac{1}{4(T+\varepsilon)} = \eta + \gamma$, we can choose γ large enough that

$$v(x, t) \leq A \exp(\alpha(|y|+r)^2) - \mu (4(\eta+\gamma))^{n/2} \exp((\eta+\gamma)r^2)$$

$$\leq \sup_{R^n} g$$

Thus let $\mu \rightarrow 0$

$$v(y, t) \leq \sup_{R^n} g$$

for all $y \in R^n$, $0 \leq t \leq T$. Now we remove the restriction on T . If eqn (1) is not true, we can simply repeatedly apply the result above on time intervals $[0, T_1]$; $[0, T_2]$, ... for $T_1 = \frac{1}{8\eta}$

Theorem: (Smoothness):- Suppose $u \in C^2(U_T)$ solves the heat equation in U_T

Then $u \in C^\infty(U_T)$

This regularity assertion is valid even if u attains non-smooth boundary value on Γ_T

Proof:- We denote the closed circular cylinder of radius r height r^2 and top center point (x, t) by

$$C(x, y; r) = \{(y, s) : |x-y| \leq r, t-r^2 \leq s \leq t\} \quad \text{--- (1)}$$

Fix $(x_0, t_0) \in U_T$. We can pick a small $r > 0$ such that

$$C = (x_0, t_0; \tau) \subset U_T$$

$$C' = C(x_0, t_0; \frac{3}{4}\tau)$$

$$C'' = C(x_0, t_0; \frac{1}{2}\tau)$$

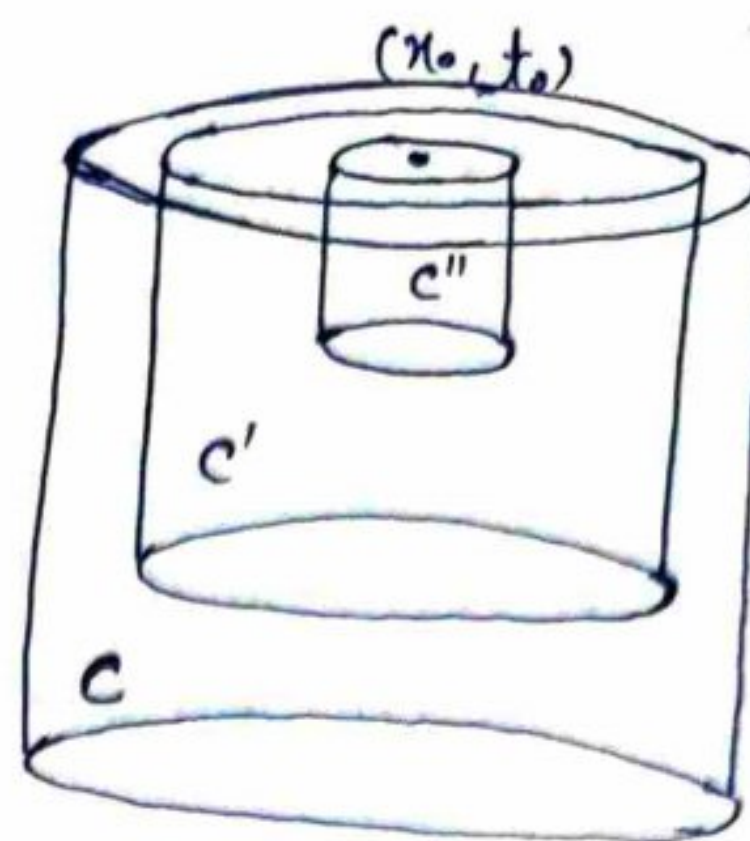
and they have same centre (x_0, t_0) .

choose a smooth cutoff function

$$\xi = \xi(x, t) \text{ with } 0 \leq \xi \leq 1$$

$$\xi \equiv 1 \text{ on } C''$$

$$\xi \equiv 0 \text{ near the parabolic boundary of } C \text{ and in } (R^n \times [0, t_0]) - C$$



We want to derive a formula of u , so we assume temporarily that $u \in C^\infty(U_T)$.

$$\text{set } v(x, t) = \xi(x, t) u(x, t) \quad (x \in R^n, 0 \leq t \leq t_0) \quad \text{--- (2)}$$

$$\text{then } v_t(x, t) = \xi_t u + \xi u_t \quad \text{--- (3)}$$

$$\Delta v = \xi \Delta u + 2 \nabla \xi \cdot \nabla u + u \Delta \xi \quad \text{--- (4)}$$

$$\text{consequently } v = 0 \text{ on } R^n \times \{t=0\} \quad \text{--- (5)}$$

Define ~~function~~ function $\bar{f}$ by

$$\bar{f} = v_t - \Delta v = \xi_t u - 2 \nabla \xi \cdot \nabla u - u \Delta \xi \text{ in } R^n \times (0, t_0)$$

$$\text{Now set } \bar{v}(x, t) = \int_0^t \int_{R^n} \Phi(x-y, t-s) \bar{f}(y, s) dy ds \quad \text{--- (6)}$$

Then $\bar{v}$ is a solution to the nonhomogeneous equation

$$\begin{cases} \bar{v}_t - \Delta \bar{v} = \bar{f} & \text{in } R^n \times (0, t_0) \\ \bar{v} = 0 & \text{on } R^n \times \{t=0\} \end{cases} \quad \text{--- (7)}$$

According to the uniqueness for Cauchy problem

$$\therefore |v| |\bar{v}| \leq A$$

$$\bar{v}(x, t) = v(x, t) = \int_0^t \int_{R^n} \Phi(x-y, t-s) \bar{f}(y, s) dy ds \quad \text{--- (8)}$$

Now suppose $(x, t) \in C''$ $\because \bar{u} \equiv 0$ in $R \times [0, t_0] - C$

(24)

$$u(x, t) = \iint_C \Phi(x-y, t-s) [\bar{u}_s(y, s) - \Delta \bar{u}(y, s) u(y, s) - 2\partial_y \bar{u}(y, s) \cdot \partial_y u(y, s)] dy ds$$

The expression in the square brackets vanishes in the region near the singularity of Φ . By integration by parts.

$$u(x, t) = \iint_C [\Phi(x-y, t-s) (\bar{u}_s(y, s) + \Delta \bar{u}(y, s)) + 2\partial_y \bar{u}(y-s, t-s) \cdot \partial_y u(y, s)] u(y, s) dy ds$$

We can derive the same formula without assuming $u \in C^\infty(U_T)$ by replacing u with $u^\epsilon = \eta_\epsilon * u$. η_ϵ is the standard mollifier in the variable x and t and let $\epsilon \rightarrow 0$ thus eqn (10) is still true if u only satisfies the hypothesis of the theorem.

Since $\bar{u} = 1$ on $C^\perp$, u has the form

$$u(x, t) = \iint_C k(x, y, t, s) u(y, s) dy ds \quad ((x, t) \in C'')$$

where $k(x, t, y, s) = 0$ for all points $(y, s) \in C''$

k is smooth on $C - C''$. From the expression of u we see that u is infinitely differentiable within

$$C'' = C(x_0, t_0; \frac{1}{2}\pi) \subset U_T$$

complete

Theorem (Backwards uniqueness):-

Suppose $u, \bar{u} \in C^2(U_T)$ solve ① & ②

$$\left. \begin{aligned} u_t - \Delta u &= 0 & \text{in } U_T \\ u &= g & \text{in on } \partial U \times [0, T] \end{aligned} \right\} \text{--- ①}$$

$$\left. \begin{aligned} \bar{u}_t - \Delta \bar{u} &= 0 & \text{in } U_T \\ \bar{u} &= g & \text{on } \partial U \times [0, T] \end{aligned} \right\} \text{--- ②}$$

for some function g . If $u(x, T) = \bar{u}(x, T) \quad (x \in U)$

then $u \equiv \bar{u}$ within U_T

Proof:- Write $w = u - \bar{u}$ and

$$\text{set } e(t) = \int_U w^2(x, t) dx \quad (0 \leq t \leq T)$$

$$\Rightarrow \begin{aligned} \dot{e}(t) &= 2 \int_U w \Delta w dx & [\because w_t - \Delta w = 0] \\ &= -2 \int_U |\nabla w|^2 dx & \text{--- ③} \end{aligned} \quad [\because -\int_U w \Delta w dx = \int_U |\nabla w|^2 dx]$$

Furthermore,

$$\begin{aligned} \ddot{e}(t) &= -4 \int_U \nabla w \cdot \nabla w_t dx \\ &= 4 \int_U \Delta w \cdot w_t dx \\ &= 4 \int_U \Delta w \cdot \Delta w dx & [\because w_t - \Delta w = 0] \\ \ddot{e}(t) &= 4 \int_U (\Delta w)^2 dx & \text{--- ④} \end{aligned}$$

Now, since $w = 0$ on ∂U

$$\int_U |\nabla w|^2 dx = -\int_U w \Delta w dx \leq \left(\int_U w^2 dx \right)^{1/2} \left(\int_U (\Delta w)^2 dx \right)^{1/2}$$

Thus from ③ & ④ implies

$$\begin{aligned} \Rightarrow [\dot{e}(t)]^2 &= 4 \left(\int_U |\nabla w|^2 dx \right)^2 \\ &\leq \left(\int_U w^2 dx \right) \left(4 \int_U (\Delta w)^2 dx \right) \\ [\dot{e}(t)]^2 &\leq e(t) \ddot{e}(t) & \text{--- ⑤} \end{aligned}$$

Now, if $e(t) = 0 \quad \forall \quad 0 \leq t \leq T$, we are done.

otherwise $\exists$ an interval $[t_1, t_2] \subset [0, T]$, with

$$e(t) > 0 \text{ for } t_1 \leq t \leq t_2, \quad e(t_2) = 0.$$

Now write $f(t) = \log e(t)$ — (7)
($t_1 \leq t \leq t_2$)

Then $\dot{f}(t) = \frac{1}{e(t)} \cdot \dot{e}(t)$

$\Rightarrow \ddot{f}(t) = \frac{e(t)\ddot{e}(t) - \dot{e}(t)\dot{e}(t)}{[e(t)]^2}$

$\Rightarrow \ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{(\dot{e}(t))^2}{(e(t))^2} \geq 0$

and so f is convex on the interval (t_1, t_2) . Consequently if

$0 < \tau < 1$, $t_1 < \tau < t_2$, we have

$f[(1-\tau)t_1 + \tau t_2] \leq (1-\tau)f(t_1) + \tau f(t_2)$ — (8)

From (7) we deduce,

$\Rightarrow e[(1-\tau)t_1 + \tau t_2] \leq e(t_1)^{(1-\tau)} \cdot e(t_2)^\tau$ — (9)

so $\log e[(1-\tau)t_1 + \tau t_2] \leq (1-\tau)\log e(t_1) + \tau \log e(t_2)$

$\Rightarrow \log e[(1-\tau)t_1 + \tau t_2] \leq \log e(t_1)^{(1-\tau)} + \log e(t_2)^\tau$

$\Rightarrow \boxed{e[(1-\tau)t_1 + \tau t_2] \leq e(t_1)^{(1-\tau)} \cdot e(t_2)^\tau}$

So, $0 \leq e[(1-\tau)t_1 + \tau t_2] \leq e(t_1)^{(1-\tau)} \cdot e(t_2)^\tau$

But in view of eqn (6) this inequality

$\Rightarrow e(t) = 0 \forall$ times $t_1 \leq t \leq t_2$ a contradiction.
[$\because e(t) \geq 0$]

complete

Wave Equation :-

The partial differential equation whose solution describe the wave propagating along some direction with some speed is known as wave equation.

The homogeneous wave equation is

$$u_{tt} - \Delta u = 0 \quad \text{--- (1)}$$

and the non-homogeneous wave equation is

$$u_{tt} - \Delta u = f \quad \text{--- (2)}$$

where $t > 0$, $x \in U$, $U \subseteq \mathbb{R}^n$ is open

The function $u: \bar{U} \times [0, \infty) \rightarrow \mathbb{R}$ is unknown.

$$\therefore u = u(x, t)$$

and Laplacian Δ is taken w.r.t spatial variables

$x = (x_1, x_2, \dots, x_n)$. In (2) the function $f: U \times [0, \infty) \rightarrow \mathbb{R}$ is given.

Remark:-

sometimes we write $u_{tt} - \Delta u = \square u$ (notation we used for wave eqn)

We shall discover that solution of the wave eqn behave quite differently than solⁿ of Laplace's eqn or the heat eqn.

for example, the solⁿ of wave eqn are generally not C^∞ .

Physical interpretation:-

The wave equation is a simplified model for a vibrating string ($n=1$), membrane ($n=2$), or elastic solid ($n=3$). In these physical interpretations $u(x, t)$ represents the displacement in some direction of the point x at time $t \geq 0$.

Let V represent any smooth subregion of U .

The acceleration within V is then

$$\frac{d^2}{dt^2} \int_V u dx = \int_V u_{tt} dx$$

and the net contact force is

$$- \int_{\partial V} F \cdot \nu \, ds$$

where F denotes the force acting on V through ∂V and the mass density is taken to be unity.

Then by Newton's Law of motion [Net force = mass $\times$ acceleration]

$$\text{i.e. } m a = F$$

$$\text{i.e. } 1 \times \int_V u_{tt} \, dx = - \int_{\partial V} F \cdot \nu \, ds$$

$$\int_V u_{tt} \, dx = - \int_{\partial V} F \cdot \nu \, ds$$

This identity obtains for each subregion V and so

$$u_{tt} = -\operatorname{div} F$$

For elastic bodies, F is a function of the displacement gradient

$$\nabla u, \text{ whence } u_{tt} + \operatorname{div} F(\nabla u) = 0$$

For small ∇u , the linearization $F(\nabla u) \approx -a \nabla u$ is often appropriate and so

$$u_{tt} - a \Delta u = 0$$

This is the wave equation if $a=1$.

* Solution for $n=1$, d' Alembert's formula:-

Initial value Problem for 1-D wave equation in all of $\mathbb{R}$ is

$$\left. \begin{aligned} u_{tt} - u_{xx} &= 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u &= g, \quad u_t = h \text{ on } \mathbb{R} \times \{t=0\} \end{aligned} \right\} \text{ --- (1)}$$

where g, h are given.

Now we have to devise a formula for u in terms of g & h .

The PDE in (1) can be factored as:-

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_t - u_{xx} = 0 \quad \text{--- (2)}$$

$$\text{Let } v(x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t) \quad \text{--- (3)}$$

$$\therefore \text{eqn (2)} \quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) v(x,t) = 0$$

$$v_t(x,t) + v_x(x,t) = 0 \quad (x \in \mathbb{R}, t > 0)$$

This is a transport eqn with constant coefficients.

$$[u_t + b u_x = 0 \Rightarrow u_t + b u_x = 0, \quad b = 1 \\ \Rightarrow u_t + u_x = 0]$$

Applying formula from initial value problem of transport equation, with $n=1, b=1$, we get

$$v(x,t) = a(x-t)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t) = a(x-t)$$

$$u_t - u_x = a(x-t) \text{ in } \mathbb{R} \times (0, \infty)$$

This is a non-homogeneous transport equation & so by eqn (3) of non-homogeneous transport equation (with $n=1, b=-1$,

$f(x,t) = a(x-t)$) implies

$$u(x,t) = \int_0^t a(x+t-s) ds + b(x+t) \quad \text{--- (4)}$$

$$= b(x+t) + \int_0^t a(x+t-2s) ds$$

$$= b(x+t) + \int_{x+t}^{x-t} -\frac{1}{2} a(y) dy$$

$$\left\{ \begin{array}{l} \text{Let } x+t-2s=y \\ -2ds=dy \\ s=0 \rightarrow y=x+t \\ s=t \rightarrow y=x-t \end{array} \right.$$

$$u(x,t) = b(x+t) + \frac{1}{2} \int_{x-t}^{x+t} a(y) dy \quad \text{--- (5)}$$

where we have $b(x) = u(x,0)$

We lastly put the initial conditions in eqn (1) to compute a & b .

The 1st initial condition in (1) gives

$$b(x) = g(x) \quad (x \in \mathbb{R})$$

where as the 2nd initial condition of (3) implies

$$a(x) = v(x,0) = u_t(x,0) - u_x(x,0)$$

$$= h(x) - g'(x) \quad (x \in \mathbb{R})$$

$\therefore$ 2nd condition is $u_t = h$ — (6)

i.e $b(x) = g(x)$ & $a(x) = h(x) - g'(x)$

putting in eqn (5)

$$\begin{aligned} u(x,t) &= g(x+t) + \frac{1}{2} \int_{x-t}^{x+t} [h(y) - g'(y)] dy \\ &= g(x+t) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy - \frac{1}{2} \int_{x-t}^{x+t} g'(y) dy \\ &= g(x+t) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy - \frac{1}{2} [g(y)]_{x-t}^{x+t} \\ &= g(x+t) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy - \frac{1}{2} g(x+t) + \frac{1}{2} g(x-t) \end{aligned}$$

$$u(x,t) = \frac{1}{2} [g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy$$

($x \in \mathbb{R}, t > 0$)

This is d'Alembert's formula.

Notation:- ① Let $x \in \mathbb{R}^n$ $t > 0, r > 0$. Define complete

$$U(x,r,t) = \int_{\partial B(x,r)} u(y,t) ds(y)$$

the average of $u(\cdot, t)$ over the sphere $\partial B(x,r)$

$$\begin{aligned} \text{② similarly } G(x,r) &= \int_{\partial B(x,r)} g(y) ds(y) \\ H(x,r) &= \int_{\partial B(x,r)} h(y) ds(y) \end{aligned} \quad \left. \vphantom{\begin{aligned} G(x,r) \\ H(x,r) \end{aligned}} \right\}$$

For fixed x , we here after regard U as a function of r & t and discover a partial differential equation U solves.

* Euler Poisson Darboux Theorem:-

statement:- for fix $x \in \mathbb{R}^n$, $n \geq 2$ and $u \in C^m(\mathbb{R}^n \times [0, \infty))$, $m \geq 1$, $n \geq 2$ be a solution of wave equation

$$\left. \begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g \text{ \& } u_t = h && \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned} \right\} \text{--- (1)}$$

where $g \in C^m(\mathbb{R}^n \times (0, \infty))$ & $h \in C^{m-1}(\mathbb{R}^n \times (0, \infty))$ are known functions

then show that $U \in C^m(\bar{\mathbb{R}}_+ \times [0, \infty))$

$$\left. \begin{aligned} \text{and } u_{tt} - u_{rr} - \frac{(n-1)}{r} u_r &= 0 && \text{in } \mathbb{R}_+ \times (0, \infty) \\ U &= G, \quad U_t = H && \text{on } \mathbb{R}_+ \times \{t=0\} \end{aligned} \right\} \text{--- (2)}$$

$$\text{i.e. } U(x, r, 0) = G(x, r), \quad U_t(x, r, 0) = H(x, r)$$

The P.D.E. in (2) is the Euler-Poisson Darboux Equation.

Note:- Here $u_{rr} + \frac{(n-1)}{r} u_r$ is the radial part of Laplacian Δ in polar coordinates.

Proof:- By using mean value (theorem) formula for Laplace eqn.

$$u_r(x, r, t) = \frac{1}{n} \int_{\partial B(x, r)} \Delta u(y, t) dy \text{ --- (3)}$$

$$\text{This implies } \lim_{r \rightarrow 0^+} u_r(x, r, t) = 0 \text{ --- (4)}$$

Differentiating eqn (3) w.r.t. r we get

$$u_{rr}(x, r, t) = \int_{\partial B(x, r)} \Delta u ds + \left(\frac{1}{n} - 1\right) \int_{B(x, r)} \Delta u dy \text{ --- (5)}$$

$$\text{Thus } \lim_{r \rightarrow 0^+} u_{rr}(x, r, t) = \frac{1}{n} \Delta u(x, t)$$

using formula (5) we can compute ~~for $r > 0$~~ u_{rrr} etc

and so verify that $U \in C^m(\bar{\mathbb{R}}_+ \times [0, \infty))$.

continuing the calculation above, we see from (2) that

$$u_r = \frac{1}{n} \int_{B(x, r)} u_{tt} dy \quad [\because u_{tt} = \Delta u]$$

$$= \frac{1}{n\alpha(n)\pi^{n-1}} \cdot \int_{\partial B(x,t)} u_H dy$$

$$\Rightarrow \pi^{n-1} U_\pi = \frac{1}{n\alpha(n)} \int_{\partial B(x,t)} u_H dy$$

$$\text{So } (\pi^{n-1} U_\pi)_\pi = \frac{1}{n\alpha(n)} \int_{\partial B(x,t)} u_H ds$$

$$= \pi^{n-1} \int_{\partial B(x,t)} u_H ds$$

$$\frac{d}{d\pi} (\pi^{n-1} U_\pi) = \pi^{n-1} U_H$$

$$\left[\begin{array}{l} \because U = \int_{\partial B(x,t)} u(y,t) ds \\ U_H = \int_{\partial B(x,t)} u_H ds \end{array} \right.$$

$$(n-1)\pi^{n-2} U_\pi + \pi^{n-1} U_{\pi\pi} = \pi^{n-1} U_H$$

$$\cancel{U_\pi} \frac{n-1}{\pi} U_\pi + U_{\pi\pi} = U_H$$

$$U_H - U_{\pi\pi} - \frac{n-1}{\pi} U_\pi = 0$$

$$\text{At } t=0, u(x,0) = g(x), u_t(x,0) = h(x)$$

$$U(x;y,t) = \int_{\partial B(x,t)} u(y,0) ds(y)$$

$$= \int_{\partial B(x,t)} g(y) ds(y)$$

$$U(x;y,0) = U(x;t)$$

$$\begin{aligned} \text{f } U_t(x;y,0) &= \int_{\partial B} u_t(y,0) ds(y) \\ &= \int_{\partial B} h(y) ds(y) \end{aligned}$$

$$U_t(x;t,0) = H(x;t) \quad \underline{\text{complete}}$$

* Solution for $n=3, 2$, Kirchhoff's & Poisson's formula

Solution for $n=3$:- Let $n=3$ and suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$
solve the initial value problem

$$\left. \begin{aligned} u_{tt} - \Delta u &= 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u &= g, \quad u_t = h \text{ in } \mathbb{R}^n \times \{t=0\} \end{aligned} \right\} \quad \text{--- (1)}$$

We Recall the Definition

$$U(x; \tau, t) = \int_{\partial B} u(y, t) ds(y) \quad \text{--- (2)}$$

$$\left. \begin{aligned} G(x; \tau) &= \int_{\partial B(x, \tau)} g(y) ds(y) \\ H(x; \tau) &= \int_{\partial B(x, \tau)} h(y) ds(y) \end{aligned} \right\} \quad \text{--- (3)}$$

and then set $\tilde{U} = \tau U$, $\tilde{G} = \tau G$, $\tilde{H} = \tau H$. --- (4)

We now assert that $\tilde{U}$ solves

$$\left. \begin{aligned} \tilde{U}_{tt} - \tilde{U}_{\tau\tau} &= 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ \tilde{U} &= \tilde{G}, \quad \tilde{U}_t = \tilde{H} \text{ on } \mathbb{R}_+ \times \{t=0\} \\ \tilde{U} &= 0 \text{ on } \{\tau=0\} \times (0, \infty) \end{aligned} \right\} \quad \text{--- (5)}$$

Indeed, $\tilde{U}_{tt} = \tau U_{tt}$

$$= \tau \left[U_{\tau\tau} + \frac{2}{\tau} U_{\tau} \right]$$

$$= \tau U_{\tau\tau} + 2U_{\tau}$$

$$\tilde{U}_{tt} = \tilde{U}_{\tau\tau}$$

[$\because n=3$ by
Euler Poisson
D'Alembert eqn]

Also at $t=0$, $U = G$ & $\therefore \tilde{U} = \tau U$

$$\Rightarrow \tilde{U} = \tau U_{t=0} = \tau G = \tilde{G}$$

$$\Rightarrow \tilde{U} = \tilde{G}$$

$$\& \tilde{U}_t = \tau U_t$$

at $t=0$

$$\tilde{U}_t = \tau U_t(t=0) = \tau H = \tilde{H} \Rightarrow \tilde{U}_t = \tilde{H}$$

& $\tilde{U} = \tau U$ at $\tau=0$ then $\tilde{U} = 0$

We find for $0 \leq \tau \leq t$

$$\tilde{U}(x; \tau, t) = \frac{1}{2} [\tilde{G}(t+\tau) - \tilde{G}(t-\tau)] + \frac{1}{2} \int_{t-\tau}^{t+\tau} \tilde{H}(y) dy \quad \text{--- (6)}$$

Since eqn (2) implies $u(x, t) = \lim_{\tau \rightarrow 0^+} U(x; \tau, t)$,

We conclude from (4) and (6), that

$$\begin{aligned} u(x, t) &= \lim_{\tau \rightarrow 0^+} \frac{\tilde{U}(x; \tau, t)}{\tau} \quad [\because \tilde{U} = \tau U] \\ &= \lim_{\tau \rightarrow 0^+} \left[\frac{\tilde{G}(t+\tau) - \tilde{G}(t-\tau)}{2\tau} + \frac{1}{2\tau} \int_{t-\tau}^{t+\tau} \tilde{H}(y) dy \right] \end{aligned}$$

$$u(x, t) = I_1 + I_2 \quad \text{--- (7)}$$

$$\begin{aligned} I_1 &= \frac{1}{2} \lim_{\tau \rightarrow 0^+} \frac{\tilde{G}(t+\tau) - \tilde{G}(t-\tau)}{\tau} = \frac{1}{2} \left[\lim_{\tau \rightarrow 0^+} \frac{\tilde{G}(t+\tau) - \tilde{G}(t)}{\tau} + \lim_{\tau \rightarrow 0^+} \frac{\tilde{G}(t) - \tilde{G}(t-\tau)}{\tau} \right] \\ &= \frac{1}{2} [\tilde{G}'(t) + \tilde{G}'(t)] \end{aligned}$$

$$I_1 = \tilde{G}'(t)$$

$$\text{Now } I_2 = \frac{1}{2} \lim_{\tau \rightarrow 0^+} \int_{t-\tau}^{t+\tau} \frac{\tilde{H}(y)}{\tau} dy = \tilde{H}(t)$$

$$\text{from eqn (7), } u(x, t) = \frac{d}{dt} \tilde{G}(x; t) + \tilde{H}(x; t) \quad \text{--- (8)}$$

$$\therefore \tilde{G}(x; t) = \tau \tilde{G}(x; \tau) = \tau \int_{\partial B(x; \tau)} g(y) ds(y)$$

$$\therefore \tilde{G}(x; t) = t \tilde{G}(x; t) = t \int_{\partial B(x; t)} g(y) ds(y)$$

$$\text{similarly } \tilde{H}(x; t) = t \tilde{H}(x; t) = t \int_{\partial B(x; t)} h(y) ds(y)$$

Therefore the solution of the wave equation in $\mathbb{R}^n$ is given

$$\text{by } u(x, t) = \frac{d}{dt} \left[t \int_{\partial B(x; t)} g(y) ds(y) \right] + t \int_{\partial B(x; t)} h(y) ds(y) \quad \text{--- (10)}$$

If g is smooth the solution can be simplified further,

In particular, for ϕ smooth, we have

$$\begin{aligned}
\frac{d}{dt} \tilde{u}(x,t) &= \frac{d}{dt} \left(t \int_{\partial B(x,t)} g(y) ds(y) \right) \\
&= \frac{d}{dt} \left(t \int_{\partial B(0,t)} g(x+tz) ds(z) \right) \\
&= \int_{\partial B(0,t)} g(x+tz) ds(z) + t \int_{\partial B(0,t)} \nabla g(x+tz) \cdot z ds(z) \\
&= \int_{\partial B(x,t)} g(y) ds(y) + t \int_{\partial B(x,t)} \nabla g(y) \left(\frac{y-x}{t} \right) ds(y) \\
&= \int_{\partial B(x,t)} g(y) ds(y) + \int_{\partial B(x,t)} \nabla g(y) (y-x) ds(y) \quad \text{--- (5)}
\end{aligned}$$

and $\tilde{H}(x,t) = t H(x,t) = t \int_{\partial B(x,t)} h(y) ds(y)$

Therefore from eqn (4), we have

$$\begin{aligned}
u(x,t) &= \int_{\partial B(x,t)} g(y) ds(y) + \int_{\partial B(x,t)} \nabla g(y) (y-x) ds(y) \\
&\quad + t \int_{\partial B(x,t)} h(y) ds(y)
\end{aligned}$$

$$u(x,t) = \int_{\partial B(x,t)} [g(y) + \nabla g(y) \cdot (y-x) + t h(y)] ds(y)$$

We note that in $\mathbb{R}^3$

$$\left[\overset{\text{volume}}{\alpha(n)} = \frac{4}{3} \pi \right]$$

$$\int_{\partial B(x,t)} = \frac{1}{n \alpha(n) t^{n-1}} \int_{\partial B(x,t)} = \frac{1}{4\pi t^2} \int_{\partial B(x,t)}$$

$\therefore$ The solution of the Initial value problem for the wave equation in $\mathbb{R}^3$ is given by

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [g(y) + \nabla g(y) \cdot (y-x) + t h(y)] ds(y)$$

This is known as Kirchhoff's formula for the solution of the initial value problem for the wave equation in $\mathbb{R}^3$

complete

* Method of descent or solution for $n=2$

Assuming $u \in C^2(\mathbb{R}^n \times [0, \infty))$ solves

$$\left. \begin{aligned} u_t - \Delta u &= 0 & x \in \mathbb{R}^n, t > 0 \\ u &= g, u_t = h \\ u &= 0 \end{aligned} \right\} \quad (1)$$

We will find a solution in the 2-D case, by using the solution to the 3-D problem.

Let $u(x_1, x_2, t)$ be the solution to the 2-D problem

define

$$\bar{u}(x_1, x_2, x_3, t) \equiv u(x_1, x_2, t) \quad (2)$$

then eqn (1) implies

$$\left. \begin{aligned} \bar{u}_t - \Delta \bar{u} &= 0 \text{ in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u} &= \bar{g}, \bar{u}_t = \bar{h} \text{ on } \mathbb{R}^3 \times \{t=0\} \end{aligned} \right\} \quad (3)$$

for $\bar{g}(x_1, x_2, x_3, 0) \equiv g(x_1, x_2, 0)$, $\bar{h}(x_1, x_2, x_3, 0) \equiv h(x_1, x_2, 0)$

If we write $x = (x_1, x_2) \in \mathbb{R}^2$ and $\bar{x} = (x_1, x_2, 0) \in \mathbb{R}^3$
then eqn (3) Kirchhoff's formula imply

$$\bar{u}(x_1, x_2, 0, t) = \int_{\partial \bar{B}(\bar{x}, t)} \left[\bar{g}(y) + \nabla \bar{g}(y) \cdot (y - \bar{x}) + t \bar{h}(y) \right] ds(y)$$

where $\bar{B}(\bar{x}, t)$ denotes the ball of radius t in $\mathbb{R}^3$ with center $\bar{x} = (x_1, x_2, 0)$. Now we note that

$$\begin{aligned} \int_{\partial \bar{B}(\bar{x}, t)} \bar{g}(y) ds(y) &= \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{x}, t)} \bar{g}(y) ds(y) \\ &= \frac{2}{4\pi t^2} \int_{B(x, t)} g(y) (1 + |D_Y(y)|^2)^{1/2} dy \end{aligned}$$

where the factor "2" enters since $\partial \bar{B}(\bar{x}, t)$ consist of two hemispheres. $B(x, t)$ is the ball in $\mathbb{R}^2$ of radius t about the point $x = (x_1, x_2)$ and $Y(y) = (t^2 - |y - x|^2)^{1/2}$

$$\therefore V(y) = (t^2 - |y-x|^2)^{1/2}$$

$$DV(y) = \frac{1}{2} \cdot \frac{1}{(t^2 - |y-x|^2)^{1/2}} \cdot 2(y-x)$$

$$DV(y) = \frac{y-x}{(t^2 - |y-x|^2)^{1/2}}$$

$$|DV(y)|^2 = \frac{|y-x|^2}{t^2 - |y-x|^2}$$

$$1 + |DV(y)|^2 = 1 + \frac{|y-x|^2}{t^2 - |y-x|^2}$$

$$1 + |DV(y)|^2 = \frac{t^2}{t^2 - |y-x|^2}$$

$$\boxed{(1 + |DV(y)|^2)^{1/2} = \left(\frac{t^2}{t^2 - |y-x|^2}\right)^{1/2}}$$

$$\therefore \int_{\partial B(\bar{x}, t)} \bar{g}(y) ds(y) = \frac{1}{2\pi t^2} \int_{B(x, t)} \frac{t g(y)}{(t^2 - |y-x|^2)^{1/2}} ds(y)$$

Similarly

$$\int_{\partial B(\bar{x}, t)} t \bar{h}(y) ds(y) = \frac{1}{2\pi t^2} \int_{B(x, t)} \frac{t^2 h(y)}{(t^2 - |y-x|^2)^{1/2}} ds(y)$$

$$\text{and } \int_{\partial B(\bar{x}, t)} \nabla \bar{g}(y) (y-x) ds(y) = \frac{1}{2\pi t^2} \int_{B(x, t)} \frac{t \nabla g(y) (y-x)}{(t^2 - |y-x|^2)^{1/2}} ds(y)$$

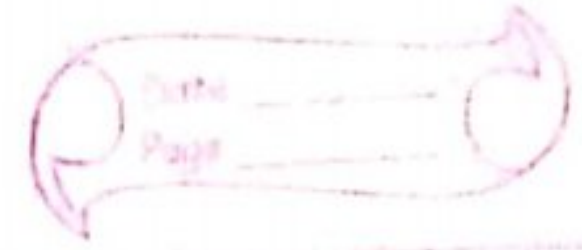
$\therefore$ the solution of the initial value problem for the wave equation in $\mathbb{R}^2$ is given by

$$\boxed{u(x, t) = \frac{1}{2\pi t^2} \int_{B(x, t)} \frac{t g(y) + t^2 h(y) + t \nabla g(y) (y-x)}{(t^2 - |y-x|^2)^{1/2}} ds(y)}$$

for $x \in \mathbb{R}^2, t > 0$. This is Poisson's formula for the solution of the initial value problem in two dimensional dimensions.

Fully completed

UNIT-II



Separation of Variables = The method of separation of Variable tries to construct a solution u to a given partial differential equation as some sort of combination of functions of few variables.

Example Let $U \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. we consider the initial boundary-Value problem for the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U \times (0, \infty) \\ u = 0 & \text{on } \partial U \times [0, \infty] \\ u = g & \text{on } \partial U \times \{t=0\} \end{cases} \quad \text{--- (i)}$$

where $g: U \rightarrow \mathbb{R}$ is given.

Solution: Let the solution of eqⁿ (i) be of the form

$$u(x, t) = V(t) \omega(x) \quad (x \in U, t \geq 0) \quad \text{--- (ii)}$$

where $x = (x_1, x_2, \dots, x_n) \in U$, $t \in [0, \tau]$

from eqⁿ (ii) $u_t(x, t) = V'(t) \omega(x)$,

$$\Delta u(x, t) = V(t) \Delta \omega(x)$$

so eqⁿ (i) reduce to

$$u_t(x,t) - \Delta u(x,t) = v'(t)w(x) - v(t)\Delta w(x) = 0$$

$$\Leftrightarrow \frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} \quad \dots (iii)$$

for all $x \in U$ and $t \geq 0$, such that $w(x), v(t) \neq 0$

Now observe that the left hand side of eqⁿ (iii) depends only on t and the right hand side depends only on x . This is impossible unless each is constant, say

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = \mu \quad (t \geq 0, x \in U)$$

$$\Rightarrow v'(t) = \mu v(t) \quad \dots (v)$$

$$\text{and } \Delta w(x) = \mu w(x) \quad \dots (vi)$$

$$\text{Now } \int \frac{v'(t)}{v(t)} dt = \mu \int dt + \log d$$

$$\log v(t) = \mu t + \log d$$

$$\Rightarrow \boxed{v(t) = d e^{\mu t}} \quad \left\{ \text{where } d \text{ is an arbitrary constant} \right\}$$

We say that λ is an eigenvalue of the operator $-\Delta$ on U provided there exists a function w not identically equal to zero, solving

$$\begin{cases} -\Delta w = \lambda w & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases} \quad \text{--- (viii)}$$

The function w is a corresponding eigen function

If λ is an eigen value and w is a related eigen function we set $u = e^{-\lambda t} w$ to find

$$u = e^{-\lambda t} w \quad \text{--- (ix) \{ by eq (ii) \}}$$

solves

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U \times (0, \infty) \\ u = 0 & \text{on } \partial U \times [0, \infty) \end{cases} \quad \text{--- (x)}$$

with the initial condition $u(\cdot, 0) = dw$ and hence from eq (ii) and eq (viii), $g = dw$

If $\lambda_1, \lambda_2, \dots, \lambda_m$ are eigenvalues, $w_1, w_2, \dots, w_m$ corresponding eigen functions and $d_1, d_2, \dots, d_m$ constants, then $u(x, t) = \sum_{k=1}^m u_k(x, t)$

$$u = \sum_{k=1}^m d_k e^{-\lambda_k t} w_k \quad \text{--- (xi)}$$

solves eq (x) with the initial condition $u(\cdot, 0) = \sum_{k=1}^m d_k w_k$ and $g = \sum_{k=1}^m d_k w_k$

Now, we can generalize

$$\sum_{k=1}^{\infty} d_k w_k = g \text{ in } U \quad \text{--- (xii)}$$

for appropriate constants $d_1, d_2, \dots$ then

$$u = \sum_{k=1}^{\infty} d_k e^{-\lambda_k t} \omega_k$$

will be the solution of the initial-value problem (1)

Answer

2018

Example. Apply the separation of variables technique to discover a solution of the porous medium equation

$$u_t - \Delta(u^r) = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \quad \text{--- (i)}$$

where $u \geq 0$ and $r > 1$ is a constant.

The expression (i) is a non-linear diffusion equation in which the rate of diffusion of some density u depends upon u itself.

Solution. Let the solution of eqⁿ (i) be the form

$$u(x, t) = v(t) \cdot w(x) \quad (x \in \mathbb{R}^n, t \geq 0) \quad \text{--- (ii)}$$

where $x = (x_1, x_2, \dots, x_n) \in U$ and $t \in [0, T]$

$$\text{Now } u_t(x, t) = v'(t) \cdot w(x)$$

$$\text{and } \Delta u(x, t) = v(t) \Delta w(x)$$

Hence from eqⁿ (i) reduce to

$$V'(t) \omega(x) - V(t)^r \Delta \omega^r(x) = 0$$

$$\Rightarrow \frac{V'(t)}{V(t)^r} = \frac{\Delta \omega^r(x)}{\omega(x)} \quad \dots (iii)$$

for all $x \in U$ and $t > 0$ such that $V(t)^r \omega(x) \neq 0$

Now from eqⁿ (iii), the LHS of eqⁿ (iii) depends only on t and the RHS of eqⁿ (iii) depends only on x . This is impossible unless each is constant say

$$\frac{V'(t)}{V(t)^r} = \frac{\Delta \omega^r(x)}{\omega(x)} = \mu \quad \dots (iv)$$

$$\Rightarrow \frac{V'(t)}{V(t)^r} = \mu \text{ and } \frac{\Delta \omega^r(x)}{\omega(x)} = \mu$$

so that $\int \frac{V'(t)}{V(t)} dt = \int \mu dt + C$

Let $V(t) = x \Rightarrow V'(t) dt = dx$

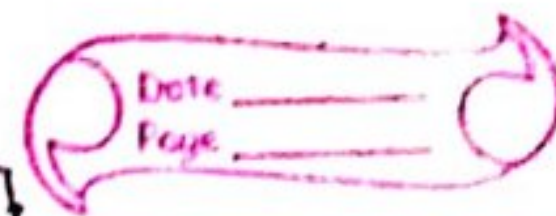
$$\therefore \int \frac{dx}{x^r} = \mu t + C$$

$$= \frac{x^{-r+1}}{-r+1} = \mu t + C$$

$$\Rightarrow x^{1-r} = (1-r)\mu t + (1-r)C$$

$$\Rightarrow (V(t))^{1-r} = (1-r)\mu t + \lambda, \quad \lambda = (1-r)C$$

$$\Rightarrow v(t) = ((1-r)ut + 1)^{\frac{1}{1-r}}$$



for some constant λ which we will take to be positive.

$$\text{and } \Delta w^r(x) = \mu w(x) \quad \dots (v)$$

Let us assume that

$$w = |x|^\alpha$$

for some constant α that must be determined.
then

$$\mu w - \Delta(w^r) = \mu |x|^\alpha - \alpha r (\alpha r + n - 2) |x|^{\alpha r - 2} \quad \dots (vi) \quad (n=1)$$

so in order that (vi) hold in $\mathbb{R}^n$ we should first require that $\alpha = \alpha r - 2$

$$\Rightarrow \alpha = \frac{2}{r-1} \quad \dots (vii)$$

from eqⁿ (vi), we see that

$$\mu = \alpha r (\alpha r + n - 2) > 0 \quad \dots (viii)$$

Hence for each $\lambda > 0$, the function

$$u = ((1-r)ut + 1)^{\frac{1}{1-r}} |x|^\alpha$$

solves the porous medium equation (i) the parameter α, μ defined by (vii), (viii).

mot
im?

Example (3) Apply the separation of Variables to discover a solution of the Hamilton-Jacobi equation

$$u_t + H(Du) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \quad \dots (i)$$

Solution. Let the solution of eqⁿ (i) be the form

$$u(x, t) = V(t) + w(x) \quad (x \in \mathbb{R}^n, t > 0)$$

Therefore, $u_t(x, t) = V'(t)$
 $H(Du(x, t)) = H(Dw(x))$

Then $0 = u_t(x, t) + H(Du(x, t)) = V'(t) + H(Dw(x))$
if and only if

$$H(Dw(x)) = \mu = -V'(t) \quad (x \in \mathbb{R}^n, t > 0)$$

for some constant μ .

$$\therefore V'(t) = -\mu$$

$$\Rightarrow V(t) = -\mu t + b$$

and if $H(Dw) = \mu$
for some $\mu \in \mathbb{R}$, then

$$u(x, t) = w(x) - \mu t + b \quad \dots (ii)$$

will for any constant b solves $u_t + H(Du) = 0$.

In particular, if we choose $w(x) = a \cdot x$ for some

$a \in \mathbb{R}^n$ and let $u = H(a)$, we discover a solution.

$u(x, t) = a \cdot x - H(a)t + b$	<u>Answer</u>
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Similarity solution.

Plane and travelling solution.

Travelling wave $\Rightarrow$ Consider a PDE involving the two variables $x \in \mathbb{R}$, $t \in \mathbb{R}$. A solution u of the form

$$u(x, t) = v(x - \sigma t) \quad (x \in \mathbb{R}, t \in \mathbb{R})$$

is called a travelling wave, (with speed σ and profile v)

plane wave: A solution u of a PDE in the $n+1$ variables $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$ having the form

$$u(x, t) = v(x \cdot y - \sigma t) \quad (x \in \mathbb{R}^n, t \in \mathbb{R})$$

is called a plane wave (with wavefront normal to $y \in \mathbb{R}^n$, velocity $\frac{\sigma}{|y|}$, and profile v).

(1.) Exponential solution $\Rightarrow$ Consider a complex Valued plane wave solutions of the form

$$u(x, t) = e^{i(y \cdot x + \omega t)} \dots (i)$$

where $\omega \in \mathbb{C}$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, ω being the frequency and $\{y_i\}_{i=1}^n$ the wave numbers.

There are three forms of exponential solution -

(a) Heat equation $\Rightarrow$ If u is given by eqⁿ (i)

i.e. $u(x, t) = e^{i(y \cdot x + \omega t)}$

then $u_t = e^{i(y \cdot x + \omega t)} \cdot \omega i = i \omega u$

$$\Delta u = \frac{d}{dx} \left(\frac{d}{dx} e^{i(y \cdot x + \omega t)} \right)$$

$$= \frac{d}{dx} (e^{i(y \cdot x + \omega t)} \cdot i y)$$

$$= i y e^{i(y \cdot x + \omega t)} \cdot i y$$

$$= -y^2 u = -|y|^2 u$$

Heat eqⁿ

$$\therefore u_t - \Delta u = (i\omega + |y|^2) u = 0$$

$$\Rightarrow i\omega + |y|^2 = 0$$

$$\Rightarrow \omega = -\frac{1}{i} |\gamma|^2 = i|\gamma|^2$$

Hence $u = e^{i\gamma x} e^{i\omega t}$
 $\Rightarrow u = e^{i\gamma x} e^{-|\gamma|^2 t}$

solves the heat equation for each $\gamma \in \mathbb{R}^n$.

Now $u = (\cos \gamma x + i \sin \gamma x) e^{-|\gamma|^2 t}$

Taking real and imaginary parts, that is

$e^{-|\gamma|^2 t} \cos(\gamma x)$ and $e^{-|\gamma|^2 t} \sin(\gamma x)$ are solutions as well.

Since ω is purely imaginary, there results a real, negative exponential term $e^{-|\gamma|^2 t}$ in the formulae, which corresponds to dissipation.

(b) Wave equation: The wave equation is given by

$$u_{tt} - \Delta u = 0$$

if $u(x, t) = e^{i(\gamma \cdot x + \omega t)}$

$$\therefore u_t = e^{i(\gamma \cdot x + \omega t)} \cdot \omega i$$

$$\Rightarrow u_{tt} = e^{i(\gamma \cdot x + \omega t)} \cdot \omega i \cdot \omega i = -\omega^2 u$$

and $\Delta u = -\gamma^2 u = -|\gamma|^2 u$

Therefore, $u_{tt} - \Delta u = (-\omega^2 + |y|^2)u = 0$

$$\Rightarrow \omega^2 = |y|^2 \Rightarrow \omega = \pm |y|$$

consequently, $u = e^{i(y \cdot x \pm |y|t)}$

Solves the wave equation, taking real and imaginary parts that is $\cos(y \cdot x \pm |y|t)$ and $\sin(y \cdot x \pm |y|t)$. Since ω is real, there are no dissipation effects in these solutions.

(c) Dispersive equations $\Rightarrow$ Let $n=1$ and substitute $u = e^{i(y \cdot x + \omega t)}$ into airy's equations

$$u_t + u_{xxx} = 0$$

So $u_t = e^{i(y \cdot x + \omega t)} i\omega = i\omega u$

And $u_x = e^{i(y \cdot x + \omega t)} iy$

$$u_{xx} = e^{i(y \cdot x + \omega t)} i^2 y^2$$

$$u_{xxx} = -iy^3 u$$

$$\therefore u_t + u_{xxx} = i(\omega - y^3)u = 0$$

$$\Rightarrow \omega - y^3 = 0 \Rightarrow \omega = y^3$$

Hence

$$u = e^{i(y \cdot x + y^3 t)}$$

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Solves Airy's equation and ω is real, so there is no dissipation.

If $n \geq 1$ and we substitute $u = e^{i(\gamma \cdot x + \omega t)}$ into Schrödinger's equation

$$iu_t + \Delta u = 0$$

$$\therefore iu_t + \Delta u = (i \cdot i\omega - |\gamma|^2)u = 0$$

$$\Rightarrow -(\omega + |\gamma|^2)u = 0$$

$$\Rightarrow \omega = -|\gamma|^2$$

Thus

$$u = e^{i(\gamma \cdot x - |\gamma|^2 t)}$$

Answer

(2) Solitons $\Rightarrow$ Consider the KdV (Korteweg-de) equation in the form.

$$u_t + 6uu_x + u_{xxx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \text{ --- (i)}$$

We seek a travelling wave solution having the form

$$u(x, t) = v(x - \sigma t) \text{ --- (ii)}$$

$$\therefore u_t = v'(x - \sigma t)(-\sigma) = \sigma v'$$

and

$$u_x = v'(x - \sigma t) = v'$$

$$u_{xx} = v''(x - \sigma t) = v''$$

from eqⁿ (i), we have

$$-\sigma V' + \sigma V V' + V''' = 0 \quad \text{--- (iii)} \quad \left\{ \frac{d}{dt} V^2 = 2V V' \right\}$$

on integrating,

$$-\sigma V + 3V^2 + V'' = a, \quad a = \text{constant} \quad \text{--- (iv)}$$

multiplying V' on both side

$$-\sigma V V' + 3V^2 V' + V' V'' = a V'$$

again integrating

$$-\frac{\sigma V^2}{2} + \frac{3V^3}{3} + \frac{(V')^2}{2} = V a + b \quad \text{--- (v)}$$

We investigate eqⁿ (v) by looking only for those solution which satisfies $V, V', V'' \rightarrow 0$ as $s \rightarrow \pm \infty$

Then eqⁿ (iv) and eqⁿ (v) implies that $a = b = 0$ so that

$$-\frac{\sigma V^2}{2} + V^3 + \frac{(V')^2}{2} = 0$$

$$\Rightarrow \frac{(V')^2}{2} = -V^3 + \frac{\sigma V^2}{2}$$

$$\Rightarrow (V')^2 = -2V^3 + \sigma V^2 = V^2 (\sigma - 2V)$$

$$\Rightarrow \boxed{V' = \pm V (\sigma - 2V)^{1/2}}$$

We take minus sign for computational convenience, and obtain then this implicit formula for V

$$S = - \int \frac{V(\theta)}{z(\sigma - 2z)^{1/2}} dz + C$$

$$\text{Let } z = \frac{\sigma}{2} \operatorname{sech}^2 \theta$$

$$\frac{dz}{d\theta} = \frac{\sigma}{2} \frac{d}{d\theta} (\cosh \theta)^{-2}$$

$$= \frac{\sigma}{2} (-2) (\cosh \theta)^{-3} (\sinh \theta)$$

$$= -\sigma \operatorname{sech}^3 \theta \sinh \theta$$

$$= -\sigma \operatorname{sech}^2 \theta \tanh \theta$$

$$\Rightarrow dz = -\sigma \operatorname{sech}^2 \theta \tanh \theta d\theta$$

$$z(\sigma - 2z)^{1/2} = \frac{\sigma}{2} \operatorname{sech}^2 \theta \cdot (\sigma - 2 \cdot \frac{\sigma}{2} \operatorname{sech}^2 \theta)^{1/2}$$

$$= \frac{\sigma}{2} \sigma^{1/2} \operatorname{sech}^2 \theta (1 - \operatorname{sech}^2 \theta)^{1/2}$$

$$= \frac{\sigma^{3/2}}{2} \operatorname{sech}^2 \theta \tanh \theta$$

$$S = - \int \frac{-\sigma \operatorname{sech}^2 \theta \tanh \theta}{\frac{\sigma^{3/2}}{2} \operatorname{sech}^2 \theta \tanh \theta} d\theta + C$$

$$= + \int \frac{2}{\sigma^{1/2}} d\theta + C$$

$$= + \frac{2\theta}{\sigma^{1/2}} + C$$

$$\Rightarrow \frac{2\theta}{\sigma^{1/2}} = S - C$$

$$\Rightarrow \theta = \frac{\sigma^{1/2} (S - C)}{2}$$

where θ is implicitly given by the relation

$$\frac{\sigma}{2} \operatorname{sech}^2 \theta = V(S)$$

$$\therefore V(s) = \frac{\sigma}{2} \operatorname{sech}^2 \left(\frac{\sigma^{1/2}}{2} (s-c) \right) \quad (s \in \mathbb{R})$$

from eqⁿ (ii), $u(x, t) = V(x - \sigma t)$

$$\Rightarrow u(x, t) = \frac{\sigma}{2} \operatorname{sech}^2 \left(\frac{\sqrt{\sigma}}{2} (x - \sigma t - c) \right)$$

is a solution of the KdV equation for each $c \in \mathbb{R}$, $\sigma > 0$.
A solution of this form is called a soliton.

NOTE: Velocity of the soliton depends upon its height
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Example. A scaling invariant solution" consider the porous medium equation.

$$u_t - \Delta(u^r) = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \quad \dots (i)$$

where $u \geq 0$ and $r > 1$ is a constant.

Solution. Let the solution be of the form

$$u(x, t) = \frac{1}{t^\alpha} V \left(\frac{x}{t^\beta} \right) \quad \dots (ii)$$

$x \in \mathbb{R}^n$, $t > 0$, where α, β are constants and $V: \mathbb{R}^n \rightarrow \mathbb{R}$ must be determined.

Now,

$$u_t = -\alpha t^{-\alpha-1} V \left(\frac{x}{t^\beta} \right) + \frac{1}{t^\alpha} DV \left(\frac{x}{t^\beta} \right) (-\beta \alpha t^{-\beta-1})$$

suppose $y = \frac{x}{t^\beta}$, then

$$u_t = -\alpha t^{-\alpha-1} v(y) - \beta t^{-\alpha-1} y Dv(y)$$

$$\text{Now } u^r(x, t) = \frac{1}{t^{\alpha r}} v^r\left(\frac{x}{t^\beta}\right)$$

$$\therefore \frac{du^r}{dx} = \frac{1}{t^{\alpha r}} (v^r)' \left(\frac{x}{t^\beta} \right) \cdot \frac{1}{t^\beta}$$

$$\Leftrightarrow \frac{d^2 u^r}{dx^2} = u^r_{xx} = \frac{1}{t^{\alpha r}} (v^r)'' \left(\frac{x}{t^\beta} \right) \cdot \frac{1}{t^{2\beta}}$$

$$= t^{-(\alpha r + 2\beta)} (v^r)''(y)$$

$$= t^{-(\alpha r + 2\beta)} (Dv^r)(y)$$

putting this value in equation (i)

$$-\alpha t^{-\alpha-1} v(y) - \beta t^{-\alpha-1} y Dv(y) - t^{-(\alpha r + 2\beta)} (Dv^r)(y) = 0$$

$$\Rightarrow \alpha t^{(-\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y Dv(y) + t^{-(\alpha r + 2\beta)} (Dv^r)(y) = 0 \quad \dots (iii)$$

In order to convert eqⁿ (iii) into an expression involving the variable y alone let us require

$$\alpha + 1 = \alpha r + 2\beta \quad \dots (*)$$

so that

$$\alpha t^{-(\alpha r + 2\beta)} v(y) + \beta t^{-(\alpha r + 2\beta)} y Dv(y) + t^{-(\alpha r + 2\beta)} (Dv^r)(y) = 0$$

$$\Rightarrow \alpha V(y) + \beta y DV(y) + (\Delta V^r)(y) = 0 \quad \dots (iv)$$

Now suppose V is radial function

$$V(y) = \omega(|y|) = \omega(r) \quad \text{for some } \omega: \mathbb{R} \rightarrow \mathbb{R}$$

$$\frac{\partial V}{\partial y_i} = \frac{\partial \omega(r)}{\partial y_i} = \frac{\partial \omega(r)}{\partial r} \frac{\partial r}{\partial y_i}$$

$$\Rightarrow \frac{\partial V}{\partial y_i} = \frac{\omega'(r) \cdot y_i}{r}$$

$$DV(y) = \frac{\omega'(r) \cdot y}{r} = \omega'(r) \quad \dots (v)$$

$$\text{and } (V^r)(y) = \omega^r(r)$$

$$\therefore \frac{dV^r}{dy_i} = \frac{d}{dy} (\omega^r(r)) = (\omega^r(r))' \frac{dr}{dy_i}$$

$$= \frac{(\omega^r(r))' y_i}{r} \quad i = 1, 2, \dots, n$$

$$\text{and } \frac{d^2}{dy_i^2} (V^r(y_i)) = \frac{d}{dy_i} \left(\frac{(\omega^r(r))' y_i}{r} \right) \quad i = 1, 2, \dots, n$$

$$\Rightarrow \Delta(V^r)(y_i) = \frac{y_i}{r} (\omega^r(r))'' \frac{dr}{dy_i} + (\omega^r(r))' \left[\frac{r - y \frac{dr}{dy}}{r^2} \right]$$

$$= \frac{(\omega^r(r))''}{r^2} \sum_{i=1}^n y_i^2 + (\omega^r(r))' \sum_{i=1}^n \left[\frac{r^2 - y_i^2}{r^3} \right]$$

$$= \frac{(\omega^r(r))''}{r} + \frac{(\omega^r(r))' (n-1)}{r} \quad \dots (vi)$$

from eqⁿ (iv), we have

$$\alpha w(r) + \beta \cdot r \cdot w'(r) + (w^r(r))'' + \frac{n-1}{r} (w^r(r))' = 0$$

set $\alpha = n\beta$, then

$$n\beta w + \beta r w' + (w^r)'' + \frac{n-1}{r} (w^r)' = 0$$

$$\Rightarrow \beta (nw + r w') + (w^r)'' + \frac{n-1}{r} (w^r)' = 0$$

$$\Rightarrow \beta (r^n w)' + (r^{n-1} (w^r)')' = 0$$

$$\Rightarrow \beta r^n w + r^{n-1} (w^r)' = a \quad (\text{on Integrating})$$

for some constant a .

Assuming $\lim_{r \rightarrow \infty} w \cdot w' = 0$ then $a = 0$

$$\text{Hence } \beta r^n w = r^{n-1} (w^r)' = 0$$

$$\Rightarrow (w^r)' = -\beta r w$$

$$\text{But then } (w^{r-1})' = -\frac{(r-1)}{r} \beta r$$

on integrating, we have

$$w^{r-1} = -\frac{(r-1)}{2r} \beta r^2 + b, \quad b \text{ is constant.}$$

$$\Rightarrow w = \left(b - \frac{(r-1)}{2r} \beta x^2 \right)^{\frac{1}{r-1}}$$

Since $V(y) = w(x)$, so from eqⁿ (ii)

$$u(x, t) = \frac{1}{t^\alpha} w\left(\frac{x}{t^\beta}\right)$$

$$\Rightarrow u(x, t) = \frac{1}{t^\alpha} \left(b - \frac{r-1}{2r} \beta \frac{|x|^2}{t^{2\beta}} \right)^{\frac{1}{r-1}} \quad (x \in \mathbb{R}^n, t > 0) \quad \dots (vii)$$

Where, since $\alpha + 1 = \alpha r + 2\beta$
and $\alpha = n\beta \Rightarrow \beta = \alpha/n$

$$\therefore \alpha + 1 = \alpha r + \frac{2\alpha}{n}$$

$$\Rightarrow n\alpha + n = \alpha nr + 2\alpha$$

$$\Rightarrow \alpha nr - n\alpha + 2\alpha = n$$

$$\Rightarrow \alpha n(r-1) + 2\alpha = n$$

$\alpha = \frac{n}{n(r-1)+2}$	and	$\beta = \frac{1}{n(r-1)+2}$	(viii)
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The formulae (vii) and eqⁿ (viii) are Barenblatt's solution to the porous medium solution.

Transform Methods $\Rightarrow$ We develop some of the theory of Fourier and Laplace transform, which provides extremely powerful tools for converting certain linear PDE into either algebraic equations or else differential equations involving fewer variables.

2012

Definition and Elementary properties $\Rightarrow$

Fourier transform on $L^1 \Rightarrow$ If $u \in L^1(\mathbb{R}^n)$ we define its Fourier transform

$$\hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot y} u(x) dx \quad (y \in \mathbb{R}^n)$$

and its inverse Fourier transform

$$\check{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot y} u(x) dx \quad (y \in \mathbb{R}^n)$$

Since $|e^{\pm ix \cdot y}| = 1$ and $u \in L^1(\mathbb{R}^n)$, these integrals converge for each $y \in \mathbb{R}^n$.

Fourier transform on $L^2 \Rightarrow$ A function $u \in L^2(\mathbb{R}^n)$, choose a sequence $\{u_k\}_{k=1}^{\infty} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

with $u_k \rightarrow u$ in $L^2(\mathbb{R}^n)$

Then

$$\|\hat{u}_k - \hat{u}_j\|_{L^2(\mathbb{R}^n)} = \|\hat{u_k - u_j}\|_{L^2(\mathbb{R}^n)} = \|u_k - u_j\|_{L^2(\mathbb{R}^n)}$$

and thus $\{\hat{u}_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$.
This sequence consequently converges to a limit,
which we define to be $\hat{u}$.

$$\hat{u}_k \rightarrow \hat{u} \text{ in } L^2(\mathbb{R}^n)$$

2020

Plancherel's Theorem ~ Assume $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then
 $\hat{u}, \check{u} \in L^2(\mathbb{R}^n)$

and

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$$

Proof - We note that if $v, w \in L^1(\mathbb{R}^n)$ then $\hat{v}, \hat{w} \in L^\infty(\mathbb{R}^n)$

$$\text{Also, } \int_{\mathbb{R}^n} v(x) \hat{w}(x) dx = \int_{\mathbb{R}^n} \hat{v}(y) w(y) dy \quad \dots (i)$$

$$\text{Now } \int_{\mathbb{R}^n} v(x) \hat{w}(x) dx = \int_{\mathbb{R}^n} \frac{v(x)}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} w(y) dy dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ixy} v(x) w(y) dx dy$$

Similarly,

$$\int_{\mathbb{R}^n} \hat{v}(x) w(x) dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ixy} v(x) w(y) dx dy$$

Furthermore,

$$\int_{\mathbb{R}^n} e^{ixy - t|x|^2} dx = \left(\frac{\pi}{t}\right)^{n/2} e^{-\frac{|y|^2}{4t}} \quad (t > 0) \quad \dots (ii)$$

Consequently, if $\epsilon > 0$ and $V_\epsilon(x) = e^{-\epsilon|x|^2}$ so that

$$\hat{V}_\epsilon(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} V_\epsilon(y) dy$$

$$\Rightarrow \hat{V}_\epsilon(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-(ixy + \epsilon|y|^2)} dy$$

$$\Rightarrow \hat{V}_\epsilon(x) = \frac{e^{-|x|^2/4\epsilon}}{(2\epsilon)^{n/2}} \quad (\text{put } x=y)$$

Thus eqⁿ (i) implies for each $\epsilon > 0$ that

$$\int_{\mathbb{R}^n} \hat{w}(y) e^{-\epsilon|y|^2} dy = \frac{1}{(2\epsilon)^{n/2}} \int_{\mathbb{R}^n} w(x) e^{-|x|^2/4\epsilon} dx \quad \text{--- (ii)}$$

Now, take $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and set $v(x) = \bar{u}(-x)$.

Let $w = u * v \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$

so that $\hat{w} = (2\pi)^{n/2} \hat{u} \cdot \hat{v} \in L^\infty(\mathbb{R}^n)$

$$\text{But } \hat{v}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} v(x) dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} \bar{u}(-x) dx$$

$$= \hat{u}(y)$$

$$\text{so } \hat{w} = (2\pi)^{n/2} \hat{u} \cdot \hat{u}$$

$$\hat{\omega} = (2\pi)^{n/2} |\hat{u}|^2 \quad \dots (iv)$$

Now $\hat{\omega}$ is continuous and thus

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{(2\varepsilon)^{n/2}} \int_{\mathbb{R}^n} \omega(x) e^{-|x|^2/4\varepsilon} dx = (2\pi)^{n/2} \omega(0) \quad \dots (v)$$

{ LHS from (iii) }

Taking $\frac{x}{2\sqrt{\varepsilon}} = z \Rightarrow dz = \frac{dx}{(2\sqrt{\varepsilon})^n}$ ($x, z \in \mathbb{R}^n$)

$$\therefore \lim_{\varepsilon \rightarrow 0} \frac{1}{(2\varepsilon)^{n/2}} \int_{\mathbb{R}^n} \omega(2\sqrt{\varepsilon}z) e^{-|z|^2} \cdot 2^n \varepsilon^{n/2} dz$$

$$= \lim_{\varepsilon \rightarrow 0} \left[2^{n/2} \int_{\mathbb{R}^n} \omega(2\sqrt{\varepsilon}z) e^{-|z|^2} dz \right]$$

$$= 2^{n/2} \omega(0) \int_{\mathbb{R}^n} e^{-|z|^2} dz$$

$$= 2^{n/2} \omega(0) \cdot (\sqrt{\pi})^n$$

$$= (2\pi)^{n/2} \omega(0)$$

so that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{(2\varepsilon)^{n/2}} \int_{\mathbb{R}^n} \omega(x) e^{-|x|^2/4\varepsilon} dx = (2\pi)^{n/2} \omega(0)$$

T. Since $\hat{\omega} = (2\pi)^{n/2} \cdot |\hat{u}|^2 \geq 0$ we deduce upon sending $\varepsilon \rightarrow 0^+$ in eqⁿ (iii) that $\hat{\omega}$ is summable, with

$$\int_{\mathbb{R}^n} \hat{\omega}(y) dy = (2\pi)^{n/2} \omega(0) \quad \{ \text{from eqⁿ (iii)} \}$$

Hence: $\int_{\mathbb{R}^n} (2\pi)^{n/2} |\hat{u}|^2 dy = (2\pi)^{n/2} \omega(0)$

$$\Rightarrow \int_{\mathbb{R}^n} |\hat{u}|^2 dy = \omega(0)$$

$$\int_{\mathbb{R}^n} |\hat{u}|^2 dy = \int_{\mathbb{R}^n} u(x) v(-x) dx$$

$$\Rightarrow \int_{\mathbb{R}^n} |\hat{u}|^2 dy = \int_{\mathbb{R}^n} u(x) \bar{u}(x) dx$$

$$\Rightarrow \int_{\mathbb{R}^n} |\hat{u}|^2 dy = \int_{\mathbb{R}^n} |u|^2 dx$$

$$\Rightarrow \left(\|\hat{u}\|_{L^2(\mathbb{R}^n)}^2 = \|u\|_{L^2(\mathbb{R}^n)}^2 \right)$$

$$\Rightarrow \|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$$

Similarly,

$$\|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$$

Hence

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$$

Proved

2018 M. Imp

Properties of Fourier transform $\Rightarrow$ Assume $u, v \in L^2(\mathbb{R}^n)$
then

$$(1.) \int_{\mathbb{R}^n} u \bar{v} dx = \int_{\mathbb{R}^n} \hat{u} \hat{\bar{v}} d$$

(2) $(\widehat{D^\alpha u}) = (iy)^\alpha \hat{u}$ for each multiindex α such that $D^\alpha u \in L^2(\mathbb{R}^n)$

(3) If $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then
 $(u \times v)^\wedge = (2\pi)^{n/2} \hat{u} \hat{v}$

(4) Furthermore $u = (\hat{u})^\vee$

Proof: (1) Let $u, v \in L^2(\mathbb{R}^n)$ and $\alpha \in \mathbb{C}$ then

$$\|u + \alpha v\|_{L^2(\mathbb{R}^n)}^2 = \|\hat{u} + \alpha \hat{v}\|_{L^2(\mathbb{R}^n)}^2$$

Expanding,

$$\int_{\mathbb{R}^n} [|u|^2 + |\alpha v|^2 + \bar{u}(\alpha v) + u(\bar{\alpha} \bar{v})] dx$$

$$= \int_{\mathbb{R}^n} [|\hat{u}|^2 + |\alpha \hat{v}|^2 + \bar{\hat{u}}(\alpha \hat{v}) + \hat{u}(\bar{\alpha} \bar{\hat{v}})] dy$$

$$\Rightarrow \|u\|_{L^2(\mathbb{R}^n)}^2 + \|\alpha v\|_{L^2(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^n} (\bar{u}(\alpha v) + u(\bar{\alpha} \bar{v})) dx$$

$$= \|\hat{u}\|_{L^2(\mathbb{R}^n)}^2 + \|\alpha \hat{v}\|_{L^2(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^n} (\bar{\hat{u}}(\alpha \hat{v}) + \hat{u}(\bar{\alpha} \bar{\hat{v}})) dy$$

$$\Rightarrow \int_{\mathbb{R}^n} (\bar{u}(\alpha v) + u(\bar{\alpha} \bar{v})) dx = \int_{\mathbb{R}^n} (\bar{\hat{u}}(\alpha \hat{v}) + \hat{u}(\bar{\alpha} \bar{\hat{v}})) dy$$

{ by Plancherel's theorem }

Taking $\alpha = 1, i$ then:

$$\int_{\mathbb{R}^n} (\bar{u}v + u\bar{v}) dx = \int_{\mathbb{R}^n} (\bar{\hat{u}}\hat{v} + \hat{u}\bar{\hat{v}}) dy \quad \text{--- (i)}$$

* and $i \int_{\mathbb{R}^n} (\bar{u} v - u \bar{v}) dx = i \int_{\mathbb{R}^n} (\hat{\bar{u}} \hat{v} - \hat{u} \hat{\bar{v}}) dy \dots (ii)$

Combining eqⁿ (i) and eqⁿ (ii), we get

$$\boxed{\int_{\mathbb{R}^n} u \bar{v} dx = \int_{\mathbb{R}^n} \hat{u} \hat{\bar{v}} dy} \quad \text{proved - I}$$

(2.) If u is smooth with compact support

$$\hat{D^\alpha u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} D^\alpha u(x) dx$$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} D_x^\alpha (e^{-ixy}) u(x) dx \quad \left\{ \begin{array}{l} \text{by integration} \\ \text{by part} \end{array} \right.$$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) \cdot (-iy)^\alpha e^{-ixy} dx$$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} (iy)^\alpha \int_{\mathbb{R}^n} u(x) e^{-ixy} dx$$

$$= (iy)^\alpha \cdot \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-ixy} dx$$

$$\Rightarrow \hat{D^\alpha u}(y) = (iy)^\alpha \hat{u}(y) \dots (iii)$$

By approximation formula eqⁿ (iii) is true if $D^\alpha u \in L^2(\mathbb{R}^n)$.

(3.) We compute for $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$.

$$(\widehat{u * v})(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} (u * v)(x) dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} \left(\int_{\mathbb{R}^n} u(z) v(x-z) dz \right) dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy - izy + izy} \left(\int_{\mathbb{R}^n} u(z) v(x-z) dz \right) dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-izy} u(z) \left(\int_{\mathbb{R}^n} e^{-i(x-z)y} v(x-z) dx \right) dz$$

$$= \int_{\mathbb{R}^n} e^{-izy} u(z) dz \widehat{v}(y) \quad \{\text{by } \widehat{v} \text{ definition}\}$$

$$\boxed{(\widehat{u * v})(y) = (2\pi)^{n/2} \widehat{u}(y) \widehat{v}(y)} \quad \left\{ \begin{array}{l} \widehat{u} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} u(x) dx \end{array} \right\}$$

Proved.

(4.) Fix $z \in \mathbb{R}^n$, $\epsilon > 0$. Let $V_\epsilon(x) = e^{i x z - \epsilon |x|^2}$

$$\therefore \widehat{V}_\epsilon(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} V_\epsilon(x) dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} e^{izx - \varepsilon|x|^2} dx$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-|y-z|^2/4\varepsilon} \left(\frac{\pi}{\varepsilon}\right)^{n/2} \quad \{\text{by result}\}$$

$$= \frac{1}{(2\varepsilon)^{n/2}} e^{-|y-z|^2/4\varepsilon}$$

Since $\int_{\mathbb{R}^n} \hat{u}(y) V_\varepsilon(y) dy = \int_{\mathbb{R}^n} u(x) \hat{V}_\varepsilon(x) dx$

$$\Rightarrow \int_{\mathbb{R}^n} \hat{u}(y) e^{iyz - \varepsilon|y|^2} dy = \int_{\mathbb{R}^n} u(x) \cdot \frac{1}{(2\varepsilon)^{n/2}} e^{-|x-z|^2/4\varepsilon} dx$$

$$\Rightarrow \int_{\mathbb{R}^n} \hat{u}(y) e^{iyz - \varepsilon|y|^2} dy = \frac{1}{(2\varepsilon)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-|x-z|^2/4\varepsilon} dx$$

The expansion of the RHE converges to $(2\pi)^{n/2} u(z)$ as $\varepsilon \rightarrow 0$

$$\therefore \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} \hat{u}(y) e^{iyz - \varepsilon|y|^2} dy = (2\pi)^{n/2} u(z)$$

$$\Rightarrow \int_{\mathbb{R}^n} \hat{u}(y) e^{iyz} dy = (2\pi)^{n/2} u(z)$$

$$\Rightarrow \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(y) e^{iyz} dy = u(z)$$

$$\Rightarrow (\hat{u}(z))^v = u(z) \quad \{\text{by definition}\}$$

$$\Rightarrow \boxed{u = (\hat{u})^v}$$

Proved

Application of Fourier Transform »

Example (1.) (Bessel potentials) » Find u

$$-\Delta u + u = f \text{ in } \mathbb{R}^n \quad \text{--- (*)}$$

where $f \in L^2(\mathbb{R}^n)$

Solution: Gives $-\Delta u + u = f \text{ in } \mathbb{R}^n \quad \text{--- (i)}$

where $f \in L^2(\mathbb{R}^n)$

Taking Fourier Transform we have

$$-\Delta \hat{u} + \hat{u} = \hat{f} \text{ in } \mathbb{R}^n \quad \text{--- (ii)}$$

using properties of Fourier Transform we have

$$D^\alpha \hat{u} = (iy)^\alpha \hat{u}, \quad D^\alpha u \in L^2(\mathbb{R}^n)$$

$$\Rightarrow \Delta \hat{u} = -|y|^2 \hat{u}$$

so from eqⁿ (ii)

$$|y|^2 \hat{u} + \hat{u} = \hat{f}$$

$$\Rightarrow \hat{u} = \frac{\hat{f}}{(1+|y|^2)}$$

suppose $\hat{B}(y) = \frac{1}{1+|y|^2}$

$$\therefore \hat{u}(y) = \hat{f}(y) \hat{B}(y)$$

$$\Rightarrow \hat{u}(y) = \frac{1}{(2\pi)^{n/2}} (f * B)^{\wedge}$$

$$\text{Now } (\hat{u}(y))^{\vee} = \left(\frac{1}{(2\pi)^{n/2}} (f * B)^{\wedge} \right)^{\vee}$$

$$\Rightarrow \boxed{u = \frac{f * B}{(2\pi)^{n/2}}} \quad \text{where } \hat{B} = \frac{1}{1+|y|^2}$$

$$\therefore u = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} B(x-y) f(y) dy \quad \text{by convolution (iii) law}$$

$$\text{Since } \int_0^{\infty} e^{-at} dt = \frac{1}{a} \quad \text{for each } a > 0$$

$$\therefore \frac{1}{1+|y|^2} = \int_0^{\infty} e^{-(1+|y|^2)t} dt$$

so that

$$B = (\hat{B})^{\vee} = \left(\frac{1}{1+|y|^2} \right)^{\vee} = \int_0^{\infty} \left(e^{-(1+|y|^2)t} \right)^{\vee} dt$$

$$= \frac{1}{(2\pi)^{n/2}} \int_0^{\infty} \left[\int_{\mathbb{R}^n} e^{ixy} e^{-(1+|y|^2)t} dy \right] dt$$

$$= \frac{1}{(2\pi)^{n/2}} \int_0^{\infty} e^{-t} \left(\int_{\mathbb{R}^n} e^{ixy - t|y|^2} dy \right) dt$$

$$\text{Since } \int_{\mathbb{R}^n} e^{ixy - t|y|^2} dy = \left(\frac{\pi}{t} \right)^{n/2} e^{-|x|^2/4t}$$

$$\therefore B = \frac{1}{(2\pi)^{n/2}} \int_0^\infty e^{-t} e^{-|x|^2/4t} \left(\frac{\pi}{t}\right)^{n/2} dt$$

$$B = \frac{1}{2^{n/2}} \int_0^\infty \frac{e^{-t - |x|^2/4t}}{t^{n/2}} dt$$

Here B is called a Bessel potential.

Now from eqⁿ (iii), we get

$$u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} B(x-y) f(y) dy$$

$$u(x) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-t - |x-y|^2/4t}}{t^{n/2}} f(y) dy dt$$

for the solution of eqⁿ (*).

Example (2.) Fundamental solution of Heat equation.

Consider the initial - Value problem for the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \quad \dots (i)$$

Solution. We establish a new method for solving eqⁿ (i) by computing $\hat{u}$.

The Fourier transform of u in the spatial variables x only. Thus

$$\begin{cases} \hat{u}_t + |y|^2 \hat{u} = c & \text{for } t > 0 \\ \hat{u} = \hat{g} & \text{for } t = 0 \end{cases} \quad \text{\{by previous\}}$$

Let $\hat{u} = v$, then

$$\frac{dv}{dt} + |y|^2 v = 0$$

$$\frac{dv}{dt} = -|y|^2 v$$

$$\frac{dv}{v} = -|y|^2 dt$$

$$\Rightarrow \log v = -|y|^2 t + \log c$$

$$\Rightarrow v = c e^{-|y|^2 t}$$

If $t = 0$, then $\hat{u} = \hat{g}$ and $v = \hat{u} = \hat{g}$

so, $\hat{g} = c$

$$\therefore v = \hat{g} e^{-|y|^2 t}$$

$$\Rightarrow \boxed{\hat{u} = \hat{g} e^{-|y|^2 t}}$$

Consequently $u = (\hat{u})^v = (\hat{g} e^{-|y|^2 t})^v$

$$\Rightarrow u = (\hat{g} \hat{F})^v \quad \text{where } \hat{F} = e^{-|y|^2 t}$$

$$\Rightarrow u = \left(\frac{(g \times F)^n}{(2\pi)^{n/2}} \right)^v = \frac{g \times F}{(2\pi)^{n/2}}$$

But $\hat{F} = e^{-|y|^2 t}$, then

$$F = (e^{-|y|^2 t})^v$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ixy} \cdot e^{-|y|^2 t} dy$$

$$= \frac{1}{(2\pi)^{n/2}} \left(\frac{\pi}{t} \right)^{n/2} e^{-|x|^2/4t}$$

i.e. $F(x) = \frac{1}{(2t)^{n/2}} e^{-|x|^2/4t}$

Therefore, $u = \frac{g \times F}{(2\pi)^{n/2}}$

$$\Rightarrow u(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} F(x-y) g(y) dy$$

$$\Rightarrow u(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{1}{(2t)^{n/2}} e^{-|x-y|^2/4t} g(y) dy$$

$$\Rightarrow u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y) dy$$

$$(x \in \mathbb{R}^n, t > 0) \quad (ii)$$

Equation (ii) is a solution for equation (i)

Example 3. (Fundamental solution of Schrödinger's equation)

Consider the initial value problem for Schrödinger's equation

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \quad (i)$$

Here u and g are complex-valued.

solution. If we replace t by it , it is fundamental solution of heat equation,

$$u(x, t) = \frac{1}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{i|x-y|^2/4t} g(y) dy \quad (i') \\ (x \in \mathbb{R}^n, t > 0)$$

$$= \frac{1}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{i|x|^2/4t} e^{i|y|^2/4t} e^{-ixy/2t} g(y) dy$$

$$= \frac{e^{i|x|^2/4t}}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy/2t} e^{i|y|^2/4t} g(y) dy$$

Since $|e^{i|x|^2/4t} \cdot e^{i|y|^2/4t}| = 1$, we can check by Plancherel's theorem, that if $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then

$$\|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} \quad (t > 0).$$

Hence the mapping $g \mapsto u(\cdot, t)$ preserves the L^2 -norm. Therefore we can extend formula (ii) to functions $g \in L^2(\mathbb{R}^n)$ in the same way that we extended the definition of Fourier transform.

Remark: we call $\psi(x, t) = \frac{1}{(4\pi it)^{n/2}} e^{i|x|^2/4t} \quad (x \in \mathbb{R}^n, t > 0)$

The fundamental solution of Schrödinger's equation

Note that formula (ii), $u = g * \psi$ for all $t \neq 0$, even $t < 0$

Thus we have solved

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \mathbb{R}^n \times (-\infty, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

In particular Schrödinger's equation is reversible in time whereas the ⁽ⁱⁱ⁾ heat equation is not.

Example 4 (Wave equation) " The initial Value problem for the wave equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \quad \text{--- (i)}$$

where for simplicity we suppose the initial velocity to be zero.

Solution: Taking the Fourier transform of u in the variable $x \in \mathbb{R}^n$, then

$$\begin{cases} \hat{u}_{tt} - \Delta \hat{u} = 0 & \text{for } t > 0 \\ \hat{u} = \hat{g}, \quad \hat{u}_t = 0 & \text{for } t = 0 \end{cases} \quad \text{--- (ii)}$$

Let $\hat{u} = v$, then

$$\begin{cases} v_{tt} - \Delta v = 0 \\ \Rightarrow v_{tt} + |y|^2 v = 0 \\ \Rightarrow (D^2 + |y|^2) v = 0 \end{cases}$$

Auxiliary equation $m^2 + |y|^2 = 0$
 $\Rightarrow m = \pm i|y|$

so $v = A \cos(|y|t) + B \sin(|y|t)$

If $t = 0$, then $v = \hat{g}$
 $\hat{g} = A + 0 \Rightarrow A = \hat{g}$

and if $t = 0$, $v_t = 0$

$$\therefore -A|y| \sin |y|t + B|y| \cos |y|t = V_t$$

$$\Rightarrow 0 + B|y| = 0 \Rightarrow B = 0$$

$$\text{Hence } \hat{u} = V = \hat{g} \cos |y|t$$

$$\Rightarrow \hat{u} = \left(\frac{e^{i|y|t} + e^{-i|y|t}}{2} \right) \hat{g}$$

$$\Rightarrow u = \left(\hat{g} \left(\frac{e^{i|y|t} + e^{-i|y|t}}{2} \right) \right)^V$$

$$\Rightarrow u(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\hat{g}(y)}{2} e^{ixy} (e^{i|y|t} + e^{-i|y|t}) dy$$

$$\Rightarrow u(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\hat{g}(y)}{2} (e^{i(xy + |y|t)} + e^{i(xy - |y|t)}) dy$$

for $x \in \mathbb{R}^n, t \geq 0$ Answer

Example(5) Telegraph equation is The initial Value problem for the One-dimensional

$$\begin{cases} u_{tt} + 2du_t - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t=0\} \end{cases} \quad (i)$$

for $d > 0$, the term " $2du_t$ " representing a physical damping of wave propagation.

Solution. Taking Fourier transform in eqⁿ (i)

$$\begin{cases} \hat{u}_{tt} + 2d\hat{u}_t - \hat{u}x^2 = 0 & \text{for } t > 0 \\ \hat{u} = \hat{g}, \quad \hat{u}_t = \hat{h} & \text{for } t = 0 \end{cases} \quad \dots (ii)$$

Let $\hat{u} = \beta e^{tr}$ ($\beta, r \in \mathbb{C}$) then

$$\hat{u}_t = r\beta e^{tr}$$

$$\Rightarrow \hat{u}_{tt} = r^2\beta e^{tr}$$

from eqⁿ (ii), we have

$$r^2\beta e^{tr} + 2dr\beta e^{tr} + |y|^2\beta e^{tr} = 0$$

$$\Rightarrow r^2 + 2dr + |y|^2 = 0$$

$$\Rightarrow r = \frac{-2d \pm \sqrt{4d^2 - 4|y|^2}}{2}$$

$$\Rightarrow r = -d \pm \frac{\sqrt{d^2 - |y|^2}}{2}$$

$$\Rightarrow r = -d \pm \sqrt{d^2 - |y|^2}$$

If $|y| \leq d$, then $(d^2 - |y|^2)^{1/2} = \xi(y)$ say

$$\therefore r(y) = -d + \xi(y) \text{ and } v(y) = -d - \xi(y)$$

If $|y| \geq d$, then $(|y|^2 - d^2)^{1/2} = \delta(y)$ (say)

$$\therefore r(y) = -d + i\delta(y) \text{ and } v(y) = -d - i\delta(y)$$

$$\text{Hence } \hat{u}(y, t) = \beta e^{rt}$$

$$\Rightarrow \hat{u}(y, t) = \begin{cases} e^{-dt} (\beta_1(y) e^{\xi(y)t} + \beta_2(y) e^{-\xi(y)t}) & \text{if } |y| \leq d \\ e^{-dt} (\beta_1(y) e^{i\delta(y)t} + \beta_2(y) e^{-i\delta(y)t}) & \text{if } |y| > d \end{cases}$$

Since $\hat{u}(y) = \hat{g}(y)$ i.e. $t=0$, then

$$\hat{u}(y, 0) = \beta_1(y) + \beta_2(y) = \hat{g}(y)$$

and

$$\hat{h}(y) = \hat{u}_t = \begin{cases} \beta_1(y) (\xi(y) - d) + \beta_2(y) (-\xi(y) - d) & \text{if } |y| \leq d \\ \beta_1(y) (i\delta(y) - d) + \beta_2(y) (-i\delta(y) - d) & \text{if } |y| > d \end{cases}$$

Since $\hat{u}(y) = \hat{g}(y)$ i.e. $t=0$ then

$$\hat{u}(y, 0) = \beta_1(y) + \beta_2(y) = \hat{g}(y)$$

Therefore,

$$u(x, t) = (\hat{u})^\vee = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ixy} \hat{u} e^{rt} dy \dots$$

$$= \frac{e^{-dt}}{(2\pi)^{n/2}} \int_{|y| \leq d} \beta_1(y) e^{ixy + \xi(y)t} + \beta_2(y) e^{ixy - \xi(y)t} dy$$

$$+ \frac{e^{-dt}}{(2\pi)^{n/2}} \int_{|y| > d} \beta_1(y) e^{i(xy + \delta(y)t)} + \beta_2(y) e^{i(xy - \delta(y)t)} dy$$

Notice the term e^{-dt} which corresponds to damping as $t \rightarrow \infty$.

Answer

2012

Laplace Transform

(Note $R_+ = (0, \infty)$)

Definition: If $u \in L^1(R_+)$ we define its Laplace transform to be

$$u^\#(s) = \int_0^\infty e^{-st} u(t) dt \quad (s \geq 0)$$

whereas the Fourier's transform is most appropriate for function defined on all of $\mathbb{R}$ (or $\mathbb{R}^n$), the Laplace transform is useful for functions defined only on R_+ .

Example (1) (Resolvents and Laplace transform)

Consider the heat equation

$$\begin{cases} V_t - \Delta V = 0 & \text{in } U \times (0, \infty) \\ V = f & \text{on } U \times \{t=0\} \end{cases} \quad (i)$$

and perform a Laplace transform w.r. to time

$$V^\#(x, s) = \int_0^\infty e^{-st} V(x, t) dt$$

$$\Delta V^\#(x, s) = \int_0^\infty e^{-st} \Delta V(x, t) dt$$

$$= \int_0^\infty e^{-st} V_t(x, t) dt \quad \text{by eq (i)}$$

$$= \left[e^{-st} V \right]_0^\infty - \int_0^\infty -s e^{-st} V(x, t) dt$$

$$= -V + \int_0^\infty s e^{-st} V(x, t) dt$$

$$= s V^{\#}(x, s) - f(x)$$

{ from eqⁿ (i) }

$$\{ e^{\infty} = \infty, e^{-\infty} = 0 \}$$

Let us write $u(x) = V^{\#}(x, s)$ for fixed $s > 0$. therefore

$$-\Delta u(x) + s u(x) = f(x) \text{ in } U \quad \dots (iii)$$

Thus the solution of the resolvent eqⁿ (iii) with RHS f of is the Laplace transform of the solution of the heat equation (i) with initial data f .

Converting Nonlinear into linear PDE:

(1) Cole-Hopf transformation:

(a) A parabolic PDE with quadratic non-linearity:

We consider on initial value problem for a quasilinear parabolic equation

$$\begin{cases} u_t - a \Delta u + b |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \quad \dots (i)$$

where $a > 0$

Solution. Assuming for the moment u is a smooth solution of eqⁿ (i), we set

$$w = \phi(u)$$

where $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function, we have

$$\omega_t = \phi'(u) u_t \quad \dots (ii)$$

and $\omega x_i = \sum_{i=1}^n \phi'(u) u x_i$

$$\Rightarrow \omega x_i x_i = \Delta \omega = \sum_{i=1}^n (\phi''(u) u^2 x_i + \phi(u) u x_i x_i)$$

$$= (\phi'(u) \Delta u + \phi''(u) |\nabla u|^2) \quad \dots (iii)$$

from eqⁿ (i) and eqⁿ (ii)

$$\begin{aligned} \omega_t &= \phi'(u) [a \Delta u - b |\nabla u|^2] \\ &= a \Delta \omega - a \phi''(u) |\nabla u|^2 - b \phi'(u) b |\nabla u|^2 \\ &= a \Delta \omega - [a \phi''(u) + b \phi'(u)] |\nabla u|^2 \\ &= a \Delta \omega \end{aligned}$$

provided we choose ϕ to satisfy $a \phi'' + b \phi' = 0$ we solve this differential equation by setting $\phi = e^{-bz/a}$ then

$$\omega = e^{-bu/a} \quad \dots (iv)$$

solves this initial-value problem for the heat equation (with conductivity a).

$$\begin{cases} \omega_t - a \Delta \omega = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ \omega = e^{-bg/a} & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \quad \dots (v)$$

Formula (iv) is the Cole-Hopf transformation

Now the unique bounded solution of eqⁿ (v) is

$$\omega(x, t) = \frac{1}{(4\pi at)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4at} e^{-b/a g(y)} dy \quad (x \in \mathbb{R}^n, t \geq 0)$$

and since eqⁿ (iv) implies

$$u = -\frac{a}{b} \log w$$

we obtain thereby the explicit formula

$$u(x, t) = -\frac{a}{b} \log \left(\frac{1}{(4\pi at)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4at} - \frac{b}{a} g(y)} dy \right) \quad (x \in \mathbb{R}^n, t > 0)$$

for a solution of quasilinear initial value problem (i).

(b) Burger's equation with viscosity $\Rightarrow$ For $n=1$, the initial value problem for the viscous Burger's equation

$$\begin{cases} u_t - au_{xx} + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t=0\} \end{cases} \quad \dots (i)$$

Solution. If we set

$$w(x, t) = \int_{-\infty}^x u(y, t) dy \quad \dots (ii)$$

$$\text{and } h(x) = \int_{-\infty}^x g(y) dy \quad \dots (iii)$$

$$\text{Therefore we have } \begin{cases} w_t - aw_{xx} + \frac{1}{2}w_x^2 = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ w = h & \text{on } \mathbb{R} \times \{t=0\} \end{cases} \quad \dots (iv)$$

$$\text{where } w_t(x, t) = \int_{-\infty}^x u_t(y, t) dy \quad \dots (v)$$

$$w_x(x, t) = \int_{-\infty}^x u_x(y, t) dy + u(y, t)$$

$$\Rightarrow w_x(x, t) = u(y, t)$$

$$\Rightarrow w_{xx}(x, t) = u_x(y, t)$$

$\left\{ \begin{array}{l} \because u \text{ is not a fun}^n \text{ of } x \end{array} \right\}$

$$\left\{ \because \frac{1}{2} w_x^2 = w_x w_{xx} \right\}$$

$$\text{Again } w_t(x, t) = \int_{-\infty}^x (a u_{yy} - u u_y) dy - u \cdot u_y \quad \{ \text{from (i)} \}$$

$$= \int_{-\infty}^x a u_{yy} dy = \int_{-\infty}^x u \cdot u_y dy$$

$$= a [u_y]_{-\infty}^x - \int_{-\infty}^x \frac{d}{dy} \left(\frac{1}{2} u^2 \right) dy$$

$$= a u_x - \left[\frac{1}{2} u^2 \right]_{-\infty}^x$$

$$= a u_x - \frac{1}{2} u^2(x)$$

$$w_t = a w_{xx} - \frac{1}{2} w_x^2$$

$$\Rightarrow \boxed{w_t - a w_{xx} + \frac{1}{2} w_x^2 = 0}$$

$$\text{Hence } \begin{cases} w_t - a w_{xx} + \frac{1}{2} w_x^2 = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ w = h & \text{on } \mathbb{R} \times \{t=0\} \end{cases} \quad \text{--- (vi)}$$

This is an equation of the form

$$\begin{cases} u_t - a \Delta u + b |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

for $n=1$, $b = \frac{1}{2}$, so that

$$w(x, t) = -2a \log \left(\frac{1}{(4\pi a t)^{1/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4at}} \frac{h(y)}{2} dy \right) \quad \text{--- (vii)}$$

differentiating w.r. to x , we have

$$u_x(x,t) = -2a \frac{1}{(4\pi at)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4at} - \frac{h(y)}{2a}} dy \left(\frac{-2(x-y)}{4at} \right)$$

$$\frac{1}{(4\pi at)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4at} - \frac{h(y)}{2a}} dy$$

$$\int_{-\infty}^{\infty} \frac{(x-y)}{t} e^{-\frac{|x-y|^2}{4at} - \frac{h(y)}{2a}} dy$$

$$u(x,t) = \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4at} - \frac{h(y)}{2a}} dy$$

is a solution of eqⁿ (i), where h is defined by eqⁿ (viii).

Potential Function: This technique uses the utilization of a potential function to convert a nonlinear system of PDE into a single linear PDE.

consider the Euler's equations for inviscid incompressible fluid flow:

$$\begin{cases} (a) & u_t + u \cdot \nabla u = -\nabla p + f & \text{in } \mathbb{R}^3 \times (0, \infty) \\ (b) & \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ (c) & u = g & \text{on } \mathbb{R}^3 \times \{t=0\} \end{cases} \quad \text{--- (i)}$$

Hence the unknown are the velocity field $u = (u^1, u^2, u^3)$ and scalar pressure p , the external force $f = (f^1, f^2, f^3)$ and initial velocity $g = (g^1, g^2, g^3)$ are given. Here ∇ the gradient in the spatial variable $x = (x_1, x_2, x_3)$. The vector equation (a) becomes

$$u_t^i + \sum_{j=1}^3 u^j u_{x_j}^i = -p_{x_i} + f^i \quad (i=1,2,3)$$

We will assume

$$\operatorname{div} u = 0 \quad \text{--- (ii)}$$

If furthermore there exists a scalar function

$h: \mathbb{R}^3 \times (0, \infty) \rightarrow \mathbb{R}$ such that

$$f = \nabla h \quad \text{--- (iii)}$$

We say that the external force is derived from the potential h .

We will try to find a solution (u, p) of (i) for which the velocity field u is also derived from a potential, say

$$u = \nabla v \quad \text{--- (iv)}$$

our flow will then be irrotational as $\operatorname{curl} u = 0$.

Now equation (i)(b)

$$0 = \operatorname{div} u = \operatorname{div}(\nabla v) = \Delta v \quad \text{--- (v)}$$

and so v must be harmonic as a function of x , for each time $t > 0$. Thus if we can find a smooth function v satisfying eqⁿ (v) and $\nabla v(\cdot, 0) = g$ hold.

We compute the pressure p . Let us observe that if $u = \nabla v$ then

$$u \cdot \nabla u = \frac{1}{2} \nabla(|\nabla v|^2) \quad \text{--- (vi)}$$

consequently (i)(b) becomes

$$\nabla_t + \frac{1}{2} \nabla(|\nabla v|^2) = -\nabla p + \nabla h$$

$$\Rightarrow D \left(V_t + \frac{1}{2} |DV|^2 \right) = D(-P+h)$$

$$\Rightarrow V_t + \frac{1}{2} |DV|^2 = -P+h \quad \dots (v_i)$$

This is Bernoulli's law.

Hodograph and Legendre transforms

(a) - Hodograph transform $\approx$ The Hodograph transform is a technique for converting certain quasilinear system of PDE into linear system by reversing the roles of the dependent and independent variables.

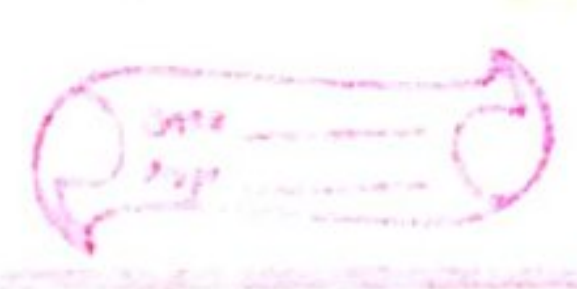
Example. The equations of steady two-dimensional irrotational fluid flow

$$\begin{cases} (a) & (\sigma^2(u) - (u^1)^2) u^1 x_1 - u^1 u^2 (u x_2^2 + u x_1^2) + (\sigma^2(u) - (u^2)^2) u x_2^2 = 0 \\ (b) & u x_2^1 - u x_1^2 = 0 \end{cases} \quad \dots (i)$$

in $\mathbb{R}^2$. The unknown is the velocity field $u = (u^1, u^2)$ and the function $\sigma(\cdot): \mathbb{R}^2 \rightarrow \mathbb{R}$, the local sound speed is given.

The eqⁿ (i) is quasilinear. let us now no longer regard u^1 and u^2 as functions of x_1 and x_2 .

$$u^1 = u^1(x_1, x_2), \quad u^2 = u^2(x_1, x_2) \quad \dots (ii)$$



but rather regard x^1 and x^2 as functions of u_1 and u_2

$$x^1 = x^1(u_1, u_2), \quad x^2 = x^2(u_1, u_2) \quad \dots (iii)$$

According to the inverse function theorem, we have

$$J = \frac{\partial(u^1, u^2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u^1}{\partial x_1} & \frac{\partial u^1}{\partial x_2} \\ \frac{\partial u^2}{\partial x_1} & \frac{\partial u^2}{\partial x_2} \end{vmatrix}$$

$$= u_{x_1}^1 u_{x_2}^2 - u_{x_2}^1 u_{x_1}^2 \neq 0$$

in some region of $\mathbb{R}^2$

$$\begin{aligned} \text{Now } du^1 &= u_{x_1}^1 dx_1 + u_{x_2}^1 dx_2 \quad \left\{ = du_1 \quad \left\{ \text{from eq}^n(ii) \right\} \right. \quad \dots (iv) \\ du^2 &= u_{x_1}^2 dx_1 + u_{x_2}^2 dx_2 \quad \left\{ = du_2 \quad \left\{ \text{from eq}^n(ii) \right\} \right. \end{aligned}$$

$$\begin{aligned} \text{and } dx^1 &= x_{u_1}^1 du_1 + x_{u_2}^1 du_2 \quad \left\{ = dx_1 \quad \left\{ \text{from eq}^n(iii) \right\} \right. \\ dx^2 &= x_{u_1}^2 du_1 + x_{u_2}^2 du_2 \quad \left\{ = dx_2 \right. \quad \dots (v) \end{aligned}$$

$$\begin{aligned} \text{from eq}^n(iv) \quad dx_1 &= \frac{du^2 u_{x_2}^1 - u_{x_2}^2 du^1}{du^1 u_{x_2}^2 - u_{x_2}^1 du^2} = \frac{-1}{-u_{x_2}^1 u_{x_2}^2 - u_{x_2}^2 u_{x_2}^1} \\ du^2 u_{x_2}^1 &= u_{x_2}^2 du^1 \end{aligned}$$

$$\Rightarrow dx_1 = \frac{-du^2 u_{x_2}^1 + u_{x_2}^2 du^1}{J} \quad \dots (vi)$$

$$\text{and } dx_2 = \frac{-u_{x_1}^2 du^1 + u_{x_1}^1 du^2}{J}$$

comparing eqⁿ (v) and eqⁿ (vi), we get

$$\left\{ \begin{array}{l} u x_2^2 = J x_{u_1}^1, \quad u x_2^1 = -J x_{u_2}^1 \\ u x_1^2 = -J x_{u_1}^2, \quad u x_1^1 = J x_{u_2}^2 \end{array} \right\} \dots (vii)$$

Insert eqⁿ (vii) into eqⁿ (i); we have

$$(a) (\sigma^2(u) - u_1^2) J x_{u_2}^2 - u_2 u_2 (-J x_{u_1}^1 - J x_{u_1}^2) + (\sigma^2(u) - u_2^2) (J x_{u_1}^1) = 0$$

$$\Rightarrow \left[(a) (\sigma^2(u) - u_1^2) x_{u_2}^2 + u_1 u_2 (x_{u_2}^1 + x_{u_1}^2) + (\sigma^2(u) - u_2^2) x_{u_1}^1 = 0 \right] \dots (viii)$$

$$(b) -J x_{u_2} + J x_{u_1}^2 = 0$$

$$\Rightarrow \boxed{x_{u_2}^1 - x_{u_1}^2 = 0} \dots (ix)$$

eqⁿ (viii) and eqⁿ (ix) is a linear system for $x = (x^0, x^1, x^2)$, as a function of $u = (u_1, u_2)$.
from book Answer

Legendre transform is The idea is to regard the component of the gradient of a solution as new independent variables.

The minimal surface equation

$$\operatorname{div} \left(\frac{Du}{(1 + |Du|^2)^{3/2}} \right) = 0$$

for $0 = 2$ may be written as

$$\operatorname{div} \left(\frac{(u_{x_1}, u_{x_2})}{(1 + u_{x_1}^2 + u_{x_2}^2)^{1/2}} \right) = 0$$

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) \left(\frac{u_{x_1}}{(1 + u_{x_1}^2 + u_{x_2}^2)^{1/2}}, \frac{u_{x_2}}{(1 + u_{x_1}^2 + u_{x_2}^2)^{1/2}} \right) = 0$$

$$\Rightarrow \frac{\partial}{\partial x_1} \left(\frac{u_{x_1}}{(1 + u_{x_1}^2 + u_{x_2}^2)^{1/2}} \right) + \frac{\partial}{\partial x_2} \left(\frac{u_{x_2}}{(1 + u_{x_1}^2 + u_{x_2}^2)^{1/2}} \right) = 0$$

$$\begin{aligned} & (1 + u_{x_1}^2 + u_{x_2}^2)^{1/2} u_{x_1 x_1} - u_{x_1} \cdot \frac{1}{2} (1 + u_{x_1}^2 + u_{x_2}^2)^{-1/2} \\ & \quad (2u_{x_1} \cdot u_{x_1 x_1} + 2u_{x_2} \cdot u_{x_1 x_2}) \\ \Rightarrow & \frac{(1 + u_{x_1}^2 + u_{x_2}^2)^{1/2} u_{x_1 x_1} - u_{x_1} \cdot \frac{1}{2} (1 + u_{x_1}^2 + u_{x_2}^2)^{-1/2} (2u_{x_1} \cdot u_{x_1 x_1} + 2u_{x_2} \cdot u_{x_1 x_2})}{(1 + u_{x_1}^2 + u_{x_2}^2)} \end{aligned}$$

$$\begin{aligned} & + \frac{(1 + u_{x_1}^2 + u_{x_2}^2)^{1/2} u_{x_2 x_2} - u_{x_2} \cdot \frac{1}{2} (1 + u_{x_1}^2 + u_{x_2}^2)^{-1/2} (2u_{x_1} \cdot u_{x_1 x_2} + 2u_{x_2} \cdot u_{x_2 x_2})}{(1 + u_{x_1}^2 + u_{x_2}^2)} = 0 \end{aligned}$$

$$\begin{aligned} & (1 + u_{x_1}^2 + u_{x_2}^2) u_{x_1 x_1} - u_{x_1} (u_{x_1} \cdot u_{x_1 x_1} + u_{x_2} \cdot u_{x_1 x_2}) \\ & + (1 + u_{x_1}^2 + u_{x_2}^2) u_{x_2 x_2} - u_{x_2} (u_{x_2} \cdot u_{x_2 x_2} + u_{x_1} \cdot u_{x_1 x_2}) = 0 \end{aligned}$$

$$\begin{aligned} & (1 + u_{x_1}^2 + u_{x_2}^2) u_{x_1 x_1} - u_{x_1}^2 u_{x_1 x_1} - u_{x_1} u_{x_2} \cdot u_{x_1 x_2} + \\ & (1 + u_{x_1}^2 + u_{x_2}^2) u_{x_2 x_2} - u_{x_2}^2 u_{x_2 x_2} - u_{x_2} u_{x_1} \cdot u_{x_1 x_2} = 0 \end{aligned}$$

$$a. (1+u_{x_2})^2 u_{x_1 x_1} - 2u_{x_1} u_{x_1 x_2} + (1+u_{x_1}^2) u_{x_2 x_2} = 0 \dots (i)$$

Let us assume that at least in some region of $\mathbb{R}^2$ we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2) \dots (ii)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2) \dots (iii)$$

The inverse function theorem, we assume that

$$J = \frac{\partial(p^1, p^2)}{\partial(x_1, x_2)} \neq 0$$

$$\Rightarrow J = \det D^2 u \neq 0 \dots (iv)$$

$$\Rightarrow \begin{vmatrix} p_{x_1}^1 & p_{x_2}^1 \\ p_{x_1}^2 & p_{x_2}^2 \end{vmatrix} \neq 0$$

$$\Rightarrow \begin{vmatrix} u_{x_1 x_1} & u_{x_1 x_2} \\ u_{x_2 x_1} & u_{x_2 x_2} \end{vmatrix} \neq 0$$

$$\Rightarrow u_{x_1 x_1} - u_{x_2 x_2} - u_{x_1 x_2} u_{x_2 x_1} \neq 0 \dots (v)$$

from eqⁿ (iii)

$$dx^1 = x_{p_1}^1 dp_1 + x_{p_2}^1 dp_2, \quad dx^2 = x_{p_1}^2 dp_1 + x_{p_2}^2 dp_2 \dots (vi)$$

from eqⁿ (ii)

$$dP^1 = U_{x_1, x_1} dx_1 + U_{x_1, x_2} dx_2, \quad dP^2 = U_{x_2, x_1} + U_{x_2, x_2} dx_2 \quad \text{--- (vi)}$$

from eqⁿ (vii)

$$\begin{aligned} & \frac{dx_1}{U_{x_1, x_2} dP^2 - U_{x_2, x_2} dP^1} = \frac{dx_2}{U_{x_1, x_1} dP^1 - U_{x_2, x_1} dP^2} \\ \Rightarrow dx_1 &= \frac{U_{x_2, x_2} dP^1 - U_{x_1, x_2} dP^2}{J}, \quad dx_2 = \frac{U_{x_1, x_1} dP^2 - U_{x_2, x_1} dP^1}{J} \quad \text{--- (viii)} \end{aligned}$$

Comparing the coefficient of dP^1 and dP^2 in eqⁿ (viii) and eqⁿ (vi)

$$\left. \begin{aligned} U_{x_2, x_2} &= J x^1_{P_1} = P^2_{x_2}, \quad U_{x_1, x_2} = -J x^1_{P_2} = P^1_{x_2} \\ \text{and } U_{x_2, x_1} &= -J x^2_{P_1} = P^2_{x_1}, \quad U_{x_1, x_1} = J x^2_{P_2} = P^1_{x_1} \end{aligned} \right\} \quad \text{--- (ix)}$$

Now we define

$$V(P) = X(P) \cdot P - U(X(P)) \quad \text{--- (x)}$$

Where $X = (x^1, x^2) = (x_1, x_2)$ and $P = (P^1, P^2) = (P_1, P_2)$

$$\text{Then } V(P_1, P_2) = (x^1_0, x^2_0) (P_1, P_2) - U(x^1_0, x^2_0)$$

$$= x^1 P_1 + x^2 P_2 - U(x^1, x^2)$$

differentiating with respect to P_2

$$V_{p_2} = p_1 x_{p_2}^1 + x^2 + p_2 x_{p_2}^2 - u_{x_1} x_{p_2}^1 - u_{x_2} x_{p_2}^2$$

$$\Rightarrow V_{p_2} = x^2 \quad \left\{ \because u_{x_1} = p_1, u_{x_2} = p_2 \right\}$$

$$\text{and } V_{p_2 p_2} = x_{p_2}^2$$

$$\Rightarrow J V_{p_2 p_2} = J x_{p_2}^2 = p_{x_1}^1 = u_{x_1} x_{x_1}^1$$

$$\Rightarrow u_{x_1} x_1 = J V_{p_2 p_2}$$

$$\text{Similarly } u_{x_1} x_2 = -J V_{p_1 p_2}$$

$$u_{x_2} x_2 = J V_{p_1 p_2}$$

----- (xi)

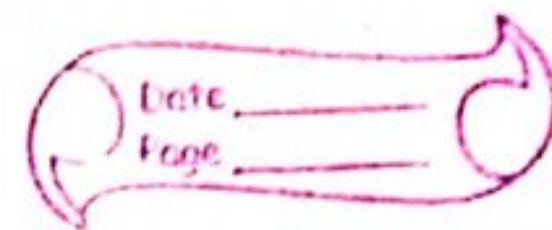
so from eqⁿ (i) and eqⁿ (xi), we get

$$(1 + p_2^2) V_{p_2 p_2} + 2 p_1 p_2 V_{p_1 p_2} + (1 + p_1^2) V_{p_1 p_1} = 0$$

Answer

UNIT-III

ASYMPTOTICS



Singular perturbations $\Rightarrow$ A singular perturbation is a modification of a given PDE by adding a small multiple ϵ times a higher order term.

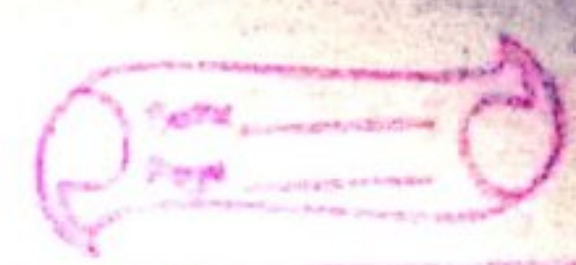
In accordance with the informal principle that the behaviour of solutions is governed by the highest order terms, a solution u^ϵ of the perturbed problem will often behave analytically quite differently from a solution u of the original equation.

Example (Transport and small diffusion) $\Rightarrow$ We study the effects of small diffusion upon the transport of dye within a moving fluid in $\mathbb{R}^2$.

Suppose a smooth vector field $b: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $b = (b^1, b^2)$ representing the steady fluid velocity. Assume dye has been continuously injected at unit rate into the fluid at the origin, and let $u(x)$ represent the density of dye at the point $x \in \mathbb{R}^2$, $x = (x_1, x_2)$. Then we have

$$\operatorname{div} \cdot (ub) = \delta_0 \text{ in } \mathbb{R}^2 \quad \text{--- (i)}$$

where δ_0 is the Dirac measure on $\mathbb{R}^2$ giving unit mass to the point 0. This PDE implies that the dye density is transported with the fluid motion of points $x \neq 0$.



consider for $\epsilon > 0$ the singular perturbation

consider for $\epsilon \rightarrow 0$ the singular per
 $-\epsilon \Delta u^\epsilon + \operatorname{div}(u^\epsilon b) = \delta_0$ in $\mathbb{R}^2$ ---- (ii)

where $\epsilon \Delta$ represents a small isotropic diffusion of the dye within the background fluid motion.

(a) Analysis of problem (i.) {eqn (i)}

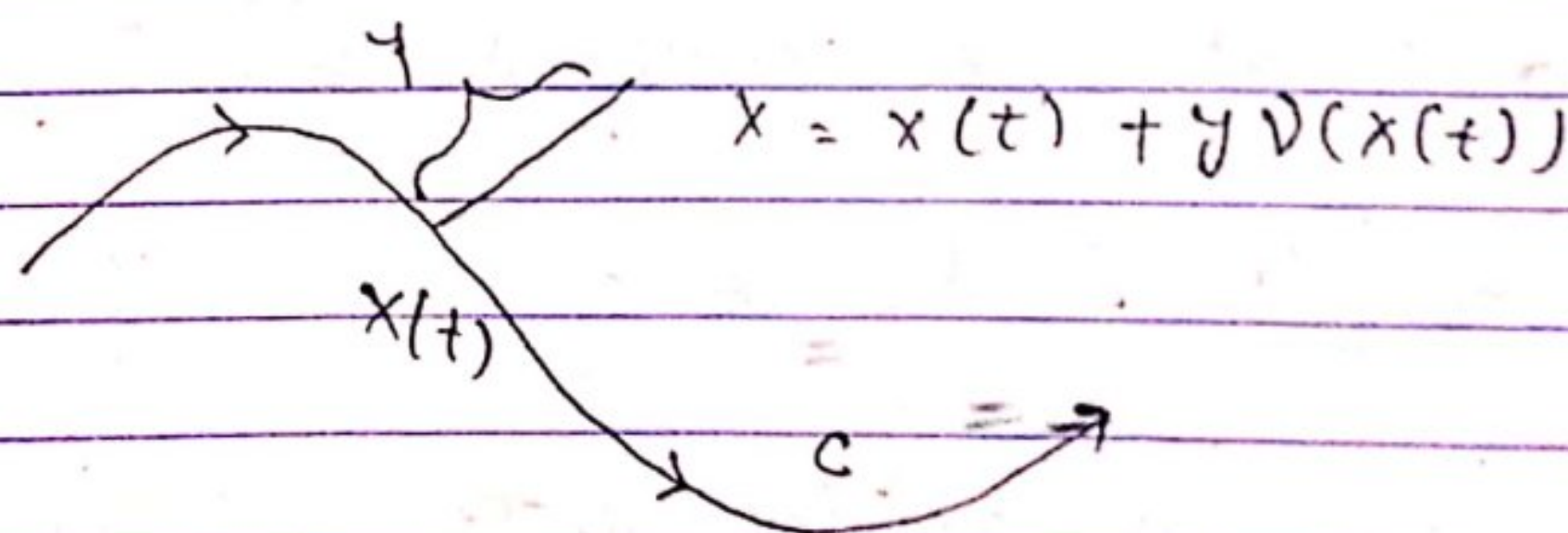
consider the unperturbed PDE

$$\operatorname{div}(ub) = \delta_0 \text{ in } \mathbb{R}^2 \text{ ---- (i)}$$

consider the characteristic ODE

$$\begin{cases} \dot{x}(t) = b(x(t)) & (t \geq 0) \\ x(0) = 0. \end{cases} \text{ ---- (ii)}$$

the solution $x(t) = (x^1(t), x^2(t))$ of which we assume to trace out a curve C , as drawn



Given a point $x \in \mathbb{R}^2$ near C , we write

$$x = x(t) = y\dot{V}(x(t)) \quad \text{--- (iii)}$$

where $\dot{V} = (\dot{V}^1, \dot{V}^2)$ is the unit normal to C , $y \in \mathbb{R}$ and t is the time required for the solution of the ODE (ii) to reach the point $x(t)$ along closest to x .

We hereafter regard (t, y) as providing a new coordinate system near the curve C , so that

$$x = (x^1(y, t), x^2(y, t))$$

Using eqⁿ (ii) and eqⁿ (iii) we compute

$$\begin{aligned} \frac{\partial (x^1, x^2)}{\partial (t, y)} &= \det \begin{pmatrix} \frac{\partial x^1}{\partial t} & \frac{\partial x^1}{\partial y} \\ \frac{\partial x^2}{\partial t} & \frac{\partial x^2}{\partial y} \end{pmatrix} \\ &= \det \begin{pmatrix} b^2 + y\dot{V}^1 & \dot{V}^1 \\ b^2 + y\dot{V}^2 & \dot{V}^2 \end{pmatrix} \\ &= b^1\dot{V}^2 + y\dot{V}^2\dot{V}^1 - (b^2\dot{V}^1 + y\dot{V}^1\dot{V}^2) \end{aligned}$$

Let us write $\sigma = |b|$, $\dot{V} = (-b^2, b^1)/\sigma$ and $\ddot{V} = -\sigma\kappa\tau = -\kappa b$ (where $\sigma = \text{speed}$, $\kappa = \text{constant}$, $\tau = b/\sigma = \text{unit tangent}$.)

Therefore, $\frac{\partial(x^1, x^2)}{\partial(t, y)} = \sigma(1 - ky) \dots (iv)$

Now we rewrite eqⁿ (i)

$$b \cdot Du + (\text{div} b)u = f_0 \text{ in } \mathbb{R}^2 \dots (v)$$

and also note that $u \equiv 0$ off the curve C .

Let u has the form

$$u(x) = p(t) \delta(y) \dots (vi)$$

in the (t, y) -coordinates, δ denoting the Dirac measure on $\mathbb{R}$ giving unit mass to the origin.

we compute,

$$\begin{aligned} \int_R u dx &= \int_{R'} p(t) \delta(y) \sigma(t) (1 - ky) dy dt \\ &= \int_{t_1}^{t_2} p(t) \sigma(t) dt \end{aligned}$$

where R is a small smooth region in the (x_1, x_2) -Plane with boundary intersecting the curve C at the points $x(t_1)$ and $x(t_2)$, $0 < t_1 < t_2$ and R' denote the corresponding region in the (t, y) -plane.

Now $\int_R u dx$ represents the total amount of dye within the region R which is to say, the total amount released between times t_1 and t_2 . This is simply $t_2 - t_1$. Thus

$$\int_{t_1}^{t_2} p(t) \sigma(t) dt = t_2 - t_1$$

This identity holds for all $0 < t_1 < t_2$ and so $\sigma(t) = (\sigma(t))'$.
from eqⁿ (vi), we have

$$u(x, t) = \delta(y) / \sigma(t) \quad \dots (vii)$$

is a solution of eqⁿ (i) for $\sigma(t) = 1/b(x(t))$, $t \geq 0$. In other words, u represents the density along the curve c of the dye, whose concentration varies inversely with the speed of the fluid.

Laplace Method $\Rightarrow$ Laplace method concerns the asymptotics as $\epsilon \rightarrow 0$ of integrals involving expressions of the form $e^{-\epsilon I}$, I denoting some given function.

09.10

Example (2.) (Vanishing viscosity method for Burger's equation):

This method investigate the limit as $\epsilon \rightarrow 0$ of the solution u^ϵ of the initial value problem for the viscous Burger's equation

$$\begin{cases} u_t^\epsilon + u^\epsilon u_x^\epsilon - \epsilon u_{xx}^\epsilon = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u^\epsilon = g & \text{on } \mathbb{R} \times \{t=0\} \end{cases} \quad \dots (i)$$

has a solution of Burger's equation is

$$u^\epsilon(x, t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{K(x, y, t)}{2\epsilon}} dy}{\int_{-\infty}^{\infty} e^{-\frac{K(x, y, t)}{2\epsilon}} dy} \quad \dots (ii)$$

$$\text{for } K(x, y, t) = \frac{|x-y|^2}{2t} + h(y) \quad (x, y \in \mathbb{R}, t > 0) \quad \dots (iii)$$

Where h is an antiderivative of g .

We investigate the limit of u^ε as $\varepsilon \rightarrow 0$. The term ' εu_x^2 ' in eqⁿ (i) makes the PDE act somewhat like the heat equation, in that the solution u^ε is infinitely differentiable in $\mathbb{R} \times (0, \infty)$ in spite of the nonlinearity. It follows from the explicit formula (ii) on the other hand, the solution u^ε should converge as $\varepsilon \rightarrow 0$ to a solution u of the conservation law

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t=0\} \end{cases} \quad \dots (iv)$$

Lemma (Asymptotics) Suppose that $k, l: \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions that l grows at most linearly and k grows at least quadratically. Assume also there exists a unique point $y_0 \in \mathbb{R}$ such that

$$k(y_0) = \min_{y \in \mathbb{R}} k(y)$$

$$\text{Then } \lim_{\varepsilon \rightarrow 0} \frac{\int_{-\infty}^{\infty} l(y) e^{-\frac{k(y)}{\varepsilon}} dy}{\int_{-\infty}^{\infty} e^{-\frac{k(y)}{\varepsilon}} dy} = l(y_0)$$

Proof. We write $k_0 = k(y_0)$. Then the function

$$\mu_\epsilon(y) = \frac{e^{\frac{\kappa_0 - \kappa(y)}{\epsilon}}}{\int_{-\infty}^{\infty} e^{\frac{\kappa_0 - \kappa(z)}{\epsilon}} dz} \quad (y \in \mathbb{R})$$

satisfies (i) $\mu_\epsilon \geq 0$

(ii) $\int_{-\infty}^{\infty} \mu_\epsilon(y) dy = 1$

(iii) $\mu_\epsilon(y) \rightarrow 0$ exponentially fast for $y \neq y_0$, as $\epsilon \rightarrow 0$

Consequently,

$$\lim_{\epsilon \rightarrow 0} \frac{\int_{-\infty}^{\infty} l(y) e^{-\frac{\kappa(y)}{\epsilon}} dy}{\int_{-\infty}^{\infty} e^{-\frac{\kappa(y)}{\epsilon}} dy} = \lim_{\epsilon \rightarrow 0} \frac{\int_{-\infty}^{\infty} l(y) e^{\frac{\kappa_0 - \kappa(y)}{\epsilon}} dy}{\int_{-\infty}^{\infty} e^{\frac{\kappa_0 - \kappa(y)}{\epsilon}} dy}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{\int_{-\infty}^{\infty} l(y) e^{\frac{\kappa_0 - \kappa(y)}{\epsilon}} dy}{\int_{-\infty}^{\infty} e^{\frac{\kappa_0 - \kappa(z)}{\epsilon}} dz}$$

$$= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} l(y) \mu_\epsilon(y) dy$$

$$= l(y_0) \quad \text{by condition?}$$

11

Geometric optics Stationary phase ~ The section investigates the behaviour of certain highly oscillatory solutions of the wave equation.

P.T.O.

1a) Geometric Optics ~

Example (Oscillating Solution) ~ Consider the wave equation

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \quad \dots (i)$$

suppose the solution $u = u^\epsilon$ of eqⁿ(i) having the form

$$u^\epsilon(x, t) = e^{\frac{i p^\epsilon(x, t)}{\epsilon}} a^\epsilon(x, t) \quad (x \in \mathbb{R}^n, t > 0)$$

the real valued function p^ϵ representing the phase and the real-valued function a^ϵ representing the amplitude. The proposed form eqⁿ(ii) for the solution is called the geometric optics ansatz*.

$$\text{Now } u_t^\epsilon = e^{i p^\epsilon / \epsilon} a_t^\epsilon + a^\epsilon e^{i p^\epsilon / \epsilon} \cdot \frac{i}{\epsilon} p_t^\epsilon$$

$$\Rightarrow u_t^\epsilon = e^{i p^\epsilon / \epsilon} \left(a_t^\epsilon + \frac{i}{\epsilon} a^\epsilon p_t^\epsilon \right)$$

$$\text{and so } u_{tt}^\epsilon = e^{i p^\epsilon / \epsilon} \left(a_{tt}^\epsilon + \frac{i}{\epsilon} (a^\epsilon p_{tt}^\epsilon + a_t^\epsilon p_t^\epsilon) \right)$$

$$+ \left(a_t^\epsilon + \frac{i}{\epsilon} (a^\epsilon p_t^\epsilon) \right) e^{i p^\epsilon / \epsilon} \cdot \frac{i}{\epsilon} p_t^\epsilon$$

$$= e^{i p^\epsilon / \epsilon} \left[\frac{a_{tt}^\epsilon + i a^\epsilon p_{tt}^\epsilon}{\epsilon} + \frac{i a_t^\epsilon p_t^\epsilon}{\epsilon} + \frac{i a_t^\epsilon p_t^\epsilon}{\epsilon} + \frac{a^\epsilon p_t^\epsilon \cdot p_t^\epsilon}{\epsilon^2} \right]$$

$$= e^{i p^\epsilon / \epsilon} \left[\frac{i p_{tt}^\epsilon}{\epsilon} a^\epsilon - \left(\frac{p_t^\epsilon}{\epsilon} \right)^2 a^\epsilon + 2 \frac{i p_t^\epsilon}{\epsilon} a_t^\epsilon + a_{tt}^\epsilon \right]$$

Again, $u_{x_i}^\epsilon = \sum_{i=1}^n \left[a^\epsilon e^{i p^\epsilon / \epsilon} \cdot \frac{i p^\epsilon}{\epsilon} + e^{i p^\epsilon / \epsilon} a_{x_i}^\epsilon \right]$

$$u_{x_i}^\epsilon = e^{i p^\epsilon / \epsilon} \sum_{i=1}^n \left[\frac{i}{\epsilon} a^\epsilon p_{x_i}^\epsilon + a_{x_i}^\epsilon \right]$$

and $u_{x_i x_i}^\epsilon = e^{i p^\epsilon / \epsilon} \sum_{i=1}^n \left[\frac{i}{\epsilon} a^\epsilon p_{x_i x_i}^\epsilon + \frac{i}{\epsilon} a_{x_i}^\epsilon p_{x_i}^\epsilon + a_{x_i x_i}^\epsilon \right]$

$$= \sum_{i=1}^n \left[\frac{i}{\epsilon} a^\epsilon p_{x_i}^\epsilon + a_{x_i}^\epsilon \right] e^{i p^\epsilon / \epsilon} \cdot \frac{i p_{x_i}^\epsilon}{\epsilon}$$

$$= e^{i p^\epsilon / \epsilon} \sum_{i=1}^n \left[\frac{i p_{x_i x_i}^\epsilon}{\epsilon} a^\epsilon - \left(\frac{p_{x_i}^\epsilon}{\epsilon} \right)^2 a^\epsilon + \frac{2 i p_{x_i}^\epsilon}{\epsilon} a_{x_i}^\epsilon + a_{x_i x_i}^\epsilon \right]$$

$$\Delta u^\epsilon = e^{i p^\epsilon / \epsilon} \left[\frac{i \Delta p^\epsilon}{\epsilon} a^\epsilon - \frac{|\Delta p^\epsilon|^2}{\epsilon^2} a^\epsilon + \frac{2 i \Delta p^\epsilon \cdot \nabla a^\epsilon}{\epsilon} + \Delta a^\epsilon \right]$$

Hence from eqⁿ (1), we get

$$u_{tt}^\epsilon - \Delta u^\epsilon = e^{i p^\epsilon / \epsilon} \left[\frac{i p_{tt}^\epsilon}{\epsilon} a^\epsilon - \left(\frac{p_t^\epsilon}{\epsilon} \right)^2 a^\epsilon + \frac{2 i p_t^\epsilon}{\epsilon} a_t^\epsilon + a_{tt}^\epsilon \right]$$

$$= e^{i p^\epsilon / \epsilon} \left[\frac{i \Delta p^\epsilon}{\epsilon} a^\epsilon - \frac{|\Delta p^\epsilon|^2}{\epsilon^2} a^\epsilon + \frac{2 i \Delta p^\epsilon \cdot \nabla a^\epsilon}{\epsilon} + \Delta a^\epsilon \right] - 0$$

Equating the real part

$$a_{tt}^\epsilon - \left(\frac{p_t^\epsilon}{\epsilon} \right)^2 a^\epsilon = \Delta a^\epsilon - \frac{|\Delta p^\epsilon|^2}{\epsilon^2} a^\epsilon$$

$$\Rightarrow a^\epsilon \left[\frac{(p_t^\epsilon)^2}{\epsilon^2} - \frac{|\Delta p^\epsilon|^2}{\epsilon^2} \right] = a_{tt}^\epsilon - \Delta a^\epsilon$$

$$\Rightarrow a^\epsilon [(p^\epsilon_t)^2 - |Dp^\epsilon|^2] = a^\epsilon (a^\epsilon_{tt} - \Delta a^\epsilon) \quad \text{--- (iii)}$$

Now if as $\epsilon \rightarrow 0$

$$p^\epsilon \rightarrow p, \quad a^\epsilon \rightarrow a \neq 0 \quad \text{--- (iv)}$$

it follows that

$$p_t^2 - |Dp_t|^2 = 0$$

$$\Rightarrow p_t \pm |Dp_t| = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

we may informally regard the straight line characteristic of these Hamiltonian Jacobi PDE as rays along which the solution (ii) concentrate in the high-frequency limit as $\epsilon \rightarrow 0$.

(b) Stationary phase :

2018.

Example. (Stationary phase for the wave equation)

The initial-value problem for the wave equation

$$\begin{cases} u^\epsilon_{tt} - \Delta u^\epsilon = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u^\epsilon = g^\epsilon, \quad u^\epsilon_t = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \quad \text{--- (i)}$$

Where, we hereafter assume g^ϵ to have the rapidly oscillating structure

$$g^\epsilon(x) = a(x) e^{iP(x)/\epsilon} \quad (x \in \mathbb{R}^n) \quad \text{--- (i)},$$

Here $\varepsilon > 0$, $a, p \in C^\infty(\mathbb{R}^n)$ and we suppose $p \neq 0$ (iii)

on the support of a .

Using Fourier transform we can write

$$u^\varepsilon(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\hat{g}^\varepsilon(y)}{2} (e^{i(x \cdot y + t|y|)} + e^{-i(x \cdot y - t|y|)}) dy \quad (x \in \mathbb{R}^n, t \geq 0)$$

Invoking eqⁿ (ii), we see

$$u^\varepsilon(x, t) = \frac{1}{2} (I_+^\varepsilon(x, t) + I_-^\varepsilon(x, t))$$

Where

$$I_\pm^\varepsilon(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(z) e^{i((x-z) \cdot y \pm t|y| + \frac{p(z)}{\varepsilon})} dy dz$$

Changing Variables gives

$$I_\pm^\varepsilon(x, t) = \frac{1}{(2\pi\varepsilon)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(z) e^{i\varepsilon\phi_\pm(x, y, z, t)} dy dz \quad \dots (iv)$$

for

$$\phi_\pm(x, y, z, t) = (x-z) \cdot y \pm t|y| + p(z) \quad \dots (v) \text{ Answer}$$

Homogenization " Homogenization theory studies the effects of high-frequency oscillations effects of high-frequency oscillations in the coefficients upon solutions of PDE. Also studies the limiting behaviour $u^\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0$.

Example. (Periodic homogenization of an elliptic equation).

Let U denote an open bounded subset of $\mathbb{R}^n$ with smooth boundary ∂U , and consider this boundary value problem for a divergence structure PDE

$$\begin{cases} -\sum_{i,j=1}^n (a^{ij}(x/\varepsilon) u_{x_i}^\varepsilon)_{x_j} = f \text{ in } U \\ u^\varepsilon = 0 \text{ in } \partial U \end{cases} \quad \dots (i)$$

Here $f: U \rightarrow \mathbb{R}$ is given and are the coefficients a^{ij} ($i, j = 1, \dots, n$) we will assume the uniform ellipticity condition

$$\sum_{i,j=1}^n a^{ij}(y) \xi_i \xi_j \geq \theta |\xi|^2$$

for some constant $\theta > 0$ and all $y, \xi \in \mathbb{R}^n$. we suppose also,

The mapping $y \mapsto a^{ij}(y)$ is $\mathbb{Q}$ -periodic. ($y \in \mathbb{R}^n$) $\dots (ii)$

$\mathbb{Q}$ denoting the unit cube in $\mathbb{R}^n$. Thus the coefficients $a^{ij}(x/\varepsilon)$ in eqⁿ (i) are rapidly oscillating in x for small $\varepsilon > 0$, and we inquire as to the effect this has upon the solution u^ε .

Let us assume $u^\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0$ $\dots (iii)$

in some suitable sense and try to determine an equation which u satisfies.

suppose u^ε admits the following two-scale expansion

$$u^\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots \quad (iv)$$

where $u_i: U \times \mathbb{Q} \rightarrow \mathbb{R}$ ($i = 0, 1, \dots$), $u_i = u_i(x, y)$

Now if $v(x) = w\left(x, \frac{x}{\varepsilon}\right)$ for some function $w = w(x, y)$
then

$$\frac{\partial v}{\partial x_i} = \frac{\partial w}{\partial x_i} + \frac{\partial w}{\partial y_i} \frac{\partial y_i}{\partial x_i} \quad \left\{ \text{let } y = \frac{x}{\varepsilon} \right\}$$

$$\frac{\partial}{\partial x_i} v = \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_i} \right) w, \quad i = 1, 2, \dots, n \quad \left\{ \frac{\partial y}{\partial x} = \frac{1}{\varepsilon} \right\}$$

Then,

$$LV = - \sum_{i,j=1}^n \left(a^{ij}\left(\frac{x}{\varepsilon}\right) v_{x_i} \right)_{x_j}$$

$$= - \sum_{i,j=1}^n \left[a^{ij}\left(\frac{x}{\varepsilon}\right) \left(\frac{\partial w}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial w}{\partial y_i} \right) \right]_{x_j}$$

$$= - \sum_{i,j=1}^n \left(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_j} \right) a^{ij}(y) \left(\frac{\partial w}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial w}{\partial y_i} \right)$$

$$= - \sum_{i,j=1}^n \left[\left(a^{ij}(y) w_{x_i} \right)_{x_j} + \left(\frac{1}{\varepsilon} a^{ij}(y) w_{x_i} \right)_{y_j} + \left(a^{ij}(y) \frac{1}{\varepsilon} w_{y_i} \right)_{x_j} + \left(\frac{1}{\varepsilon^2} a^{ij}(y) w_{y_i} \right)_{y_j} \right]$$

$$LV = \frac{1}{\epsilon^2} L_1 \omega + \frac{1}{\epsilon} L_2 \omega + L_3 \omega \dots (v)$$

where

$$\begin{cases} (a) L_1 \omega = - \sum_{i,j=1}^n (a^{ij}(y) \omega y_i) y_j \\ (b) L_2 \omega = - \sum_{i,j=1}^n (a^{ij}(y) \omega x_i) y_j + (a^{ij}(y) \omega x_i) g x_j \\ (c) L_3 \omega = - \sum_{i,j=1}^n (a^{ij}(y) \omega x_i) x_j \end{cases}$$

Putting this value in eqⁿ (i)

$$L u^\epsilon(x) = f$$

$$\Rightarrow \frac{1}{\epsilon^2} L_1 u^\epsilon(x) + \frac{1}{\epsilon} L_2 u^\epsilon(x) + L_3 u^\epsilon(x) = f$$

$$\Rightarrow \frac{1}{\epsilon^2} L_1 [u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots] + \frac{1}{\epsilon} L_2 [u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots] + L_3 [u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots] = f$$

$$\Rightarrow \frac{1}{\epsilon^2} L_1 u_0 + \frac{1}{\epsilon} (L_1 u_1 + L_2 u_0) + (L_1 u_2 + L_2 u_1 + L_3 u_0) + \{ \text{terms involving } \epsilon, \epsilon^2, \dots \} = f$$

Equating like powers of ϵ , we deduce

$$\begin{cases} (a) L_1 u_0 = 0 \\ (b) L_1 u_1 + L_2 u_0 = 0 \\ (c) L_1 u_2 + L_2 u_1 + L_3 u_0 = f \text{ etc} \end{cases} \dots (vi)$$

we examine these PDE to deduce information concerning $u_0, u_1, u_2, \dots$. Now in view of (5a), (7a) for each (v)(a), (vii)(a) for each fixed x .

$u_0(x, y)$ solves $L_1 u_0 = 0$ and is $\mathbb{Q}$ -periodic.

It turns out that the only such solutions are constant in y . This is fact

$u_0 = u(x)$ depends only on x ---- (viii)

so from (vi)(b) and (vii)(b) we obtain

$$L_1 u_1 = \sum_{i,j=1}^n a^{ij}(y) y_j u_{x_i} \text{ ---- (ix)}$$

For $i=1, 2, \dots, n$, let $X^i = X^i(y)$, solve

$$\begin{cases} L_1 X^i = - \sum_{j=1}^n a^{ij}(y) y_j & \text{in } \mathbb{Q} \\ X^i & \mathbb{Q} \text{-periodic} \end{cases} \text{ ---- (x)}$$

as the RHS of the PDE in eqⁿ(x) has integral zero over $\mathbb{Q}$, this problem has a solution X^i .

$$\text{Now } u_1(x, y) = - \sum_{i=1}^n X^i(y) u_{x_i}(x) + \tilde{u}_1(x) \text{ ---- (*)}$$

where $\tilde{u}_1$ denoting an arbitrary function of x alone.

from (vii)(c)

$$L_1 u_2 = f - L_2 u_1 - L_3 u_0 \dots (xi)$$

This PDE will have a $\mathbb{Q}$ -periodic solution only if the integral of the RHS over $\mathbb{Q}$ is zero.

Thus we require

$$\int_{\mathbb{Q}} (L_2 u_1 + L_3 u_0) dy = \int_{\mathbb{Q}} f dy = f(x) \dots (xii)$$

from eqⁿ (vi)(b) and (*)

$$\begin{aligned} \int_{\mathbb{Q}} L_2 u_1 dy &= - \sum_{j,k=1}^n \left(\int_{\mathbb{Q}} a^{jk}(y) u_1 y_k dy \right)_{x_j} \\ &= - \sum_{j,k=1}^n \left(\int_{\mathbb{Q}} a^{jk}(y) \chi_{y_k}^i(y) dy \right) u_{x_i x_j} \end{aligned}$$

Since $u = u_0$, this calculation and eqⁿ (xii) imply

$$- \sum_{i,j=1}^n \left(\int_{\mathbb{Q}} a^{ij}(y) - \sum_{k=1}^n a^{jk}(y) \chi_{y_k}^i(y) dy \right) u_{x_i x_j}(x) = f(x)$$

i.e.

$$\begin{cases} - \sum_{i,j=1}^n a^{-ij} u_{x_i x_j} = f \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases} \dots (xiii)$$

$$\text{where } a^{-ij} = \int_{\mathbb{Q}} a^{ij}(y) - \sum_{k=1}^n a^{jk}(y) \chi_{y_k}^i(y) dy \quad (i, j = 1, 2, \dots, n)$$

are the homogenized coefficients and χ^i solves the corrector problem (x) ($i=1, \dots, n$). Thus we expect $u^\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0$ and $u \neq 0$

Solve the limit problem (xiii).

Power series -

2018

Non-characteristic surface = The solvability of the k^{th} -order quasilinear

PDE

$$\sum_{|\alpha|=k} a_{\alpha} (D^{k-1}u, D^{k-2}u, \dots, u, x) D^{\alpha}u + a_0 (D^{k-1}u, \dots, u, x) = 0$$

in some open region $U \subset \mathbb{R}^n$. Let us assume that Γ is a smooth $(n-1)$ -dimensional hypersurface in U . the unit normal to which at any point $x^0 \in \Gamma$ is $\nabla(x^0) = \nabla = (\nabla_1, \nabla_2, \dots, \nabla_n)$.

Notation: The j^{th} normal derivative of u at $x^0 \in \Gamma$ is

$$\frac{\partial^j u}{\partial \nabla^j} = \sum_{|\alpha|=j} D^{\alpha}u \nabla^{\alpha} = \sum_{\alpha_1+\alpha_2+\dots+\alpha_n=j} \frac{\partial^j u}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n} \nabla^{\alpha_1} \dots \nabla^{\alpha_n}$$

Now let $g_0, \dots, g_{k-1} : \Gamma \rightarrow \mathbb{R}$ be k given functions.

The Cauchy problem is then to find a function u solving the PDE (i) subject to the boundary condition

$$u = g_0, \frac{\partial u}{\partial \nabla} = g_1, \frac{\partial^2 u}{\partial \nabla^2} = g_2, \dots, \frac{\partial^{k-1} u}{\partial \nabla^{k-1}} = g_{k-1} \text{ on } \Gamma \quad \text{--- (ii)}$$

We say that the eqⁿ (ii) prescribe the Cauchy data $g_0, g_1, \dots, g_{k-1}$ on Γ .

General surfaces

Definition: We say the surface Γ is noncharacteristic for the PDE.

$$\sum_{|\alpha|=k} a_\alpha (D^{k-1} u, \dots, u, x) D^\alpha u + a_0 (D^{k-1} u, \dots, u, x) = 0$$

provided $\sum_{|\alpha|=k} a_\alpha D^\alpha \neq 0$ on Γ

for all values of the arguments of the coefficients a_α ($|\alpha|=k$).

Real Analytic function $\Rightarrow$ A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is called (Real) Analytic near x_0 if there exists $\delta > 0$ and constant $\{f_\alpha\}$ such that

$$f(x) = \sum_{\alpha} f_\alpha (x - x_0)^\alpha \quad (|x - x_0| < \delta)$$

The sum taken over all multiindices α .

Remark (i) We write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ for the multiindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$

(ii) If f is analytic near x_0 , then f is C^∞ near x_0

the constants f_α are computed as $f_\alpha = \frac{D^\alpha f(x_0)}{\alpha!}$.

where $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$

Thus f equals its Taylor expansion about x_0

$$f(x) = \sum_{\alpha} \frac{1}{\alpha!} D^\alpha f(x_0) (x - x_0)^\alpha \quad (|x - x_0| < \theta \varepsilon)$$

Majorizer Let $f = \sum_{\alpha} f_{\alpha} x^{\alpha}$, $g = \sum_{\alpha} g_{\alpha} x^{\alpha}$ be two power series. We say g majorizes f , written $g \gg f$ provided.

$$g_{\alpha} \geq |f_{\alpha}| \quad \text{for all multiindices } \alpha.$$

Lemma (Majorants).

(i) If $g \gg f$, and g converges for $|x| < r$, then f also converges for $|x| < r$.

(ii) If $f = \sum_{\alpha} f_{\alpha} x^{\alpha}$ converges for $|x| < r$ and $0 < s/\sqrt{n} < r$, then f has a majorant for $|x| < s/\sqrt{n}$.

Proof. (i) To verify (i), we check

$$\sum_{\alpha} |f_{\alpha} x^{\alpha}| \leq \sum_{\alpha} g_{\alpha} |x_1|^{\alpha_1} \dots |x_n|^{\alpha_n} < \infty \quad \text{if } |x| < r$$

$\Rightarrow f$ converges for $|x| < r$.

(2) Let $0 < s\sqrt{n} < r$ and set $y = s(1, 1, \dots, 1)$ then

$|y| = s\sqrt{n} < r$ and so $\sum f_\alpha y^\alpha$ converges. Thus there exists a constant c such that

$$|f_\alpha y^\alpha| \leq c \text{ for each multiindex } \alpha.$$

In particular,

$$|f_\alpha| \leq \frac{c}{y_1^{\alpha_1} \dots y_n^{\alpha_n}} = \frac{c}{s^{|\alpha|}} \leq \frac{c |\alpha|!}{s^{|\alpha|} \alpha!}$$

But then

$$g(x) = \frac{Cs}{s - (x_1 + \dots + x_n)} = c \sum_{\alpha} \frac{|\alpha|!}{s^{|\alpha|} \alpha!} x^\alpha$$

majorizes f for $|x| < s/\sqrt{n}$ Proved

20.2

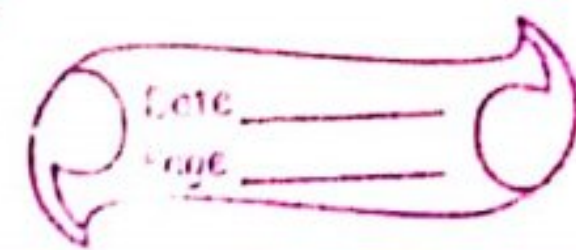
Cauchy - Kovalevskaya Theorem Assume $\{B_j\}_{j=1}^{n-1}$ and c are real analytic function. Then there exist $r > 0$ and a real analytic function

$$u = \sum_{\alpha} u_{\alpha} x^{\alpha} \quad \dots (i)$$

Solving the boundary Value problem

$$\begin{cases} u_{x_n} = \sum_{j=1}^{n-1} B_j(u, x') u_{x_j} + c(u, x') & \text{for } |x'| < r \quad \dots (ii) \\ u = 0 & \text{for } |x'| < r, x_n = 0 \end{cases}$$

where the analytic function $B_j: \mathbb{R}^m \times \mathbb{R}^{m-1} \rightarrow \mathbb{R}^{m \times m}$



($j=1, 2, \dots, n-1$) and $C: \mathbb{R}^m \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^m$ is given.

Proof. We must compute the coefficients

$$u_\alpha = \frac{D^\alpha u(0)}{\alpha!} \quad \dots (iii)$$

in terms of $\{B_j\}_{j=1}^{n-1}$ and c and then show that the power series (iii), so obtained in fact converges if $|x| < \sigma$ and $\sigma > 0$ is small enough.

The functions $\{B_j\}_{j=1}^{n-1}$ and c are analytic we can write

$$B_j(z, x') = \sum_{r,s} B_{j,r,s} z^r x'^s \quad (j=1, \dots, n-1) \quad \dots (iv)$$

$$\text{and} \quad c(z, x') = \sum_{r,s} C_{r,s} z^r x'^s \quad \dots (v)$$

These power series converge if $|z| + |x'| < \delta$ for some small $\delta > 0$. Thus

$$B_{j,r,s} = \frac{D_z^r D_{x'}^s B_j(0,0)}{(r+s)!}, \quad C_{r,s} = \frac{D_z^r D_{x'}^s c(0,0)}{(r+s)!} \quad \dots (vi)$$

for $j=1, 2, \dots, n-1$ and all multiindices r, s

since $u \equiv 0$ on $\{x_n = 0\}$, we have

$$u_\alpha = \frac{D^\alpha u(0)}{\alpha!} = 0 \quad \text{for all multiindices } \alpha \text{ with } \alpha_n = 0 \quad \dots (vii)$$

Now fix $i \in \{1, \dots, n-1\}$ and differentiating eqⁿ (ii') with respect to x_i

$$G \quad u^k_{x_n x_i} = \sum_{j=1}^{n-1} \sum_{l=1}^m \left(b_j^{kl} u^l_{x_i x_j} + b_j^{kl} u^l_{x_i} + \sum_{p=1}^m b_j^{kl} z_p u^p_{x_i} u^l_{x_j} \right. \\ D \quad \left. + c_{x_i}^k + \sum_{p=1}^m z_p^k u^p_{x_i} \right)$$

from eqⁿ (vii) we conclude that $u^k_{x_n x_i}(0) = c_{x_i}^k(0,0)$

If α is a multiindex having the form $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n-1}, 2)$
 $= (\alpha', 2)$

we prove by induction that

$$D^\alpha u^k(0) = D^{\alpha'} e^k(0,0)$$

Next suppose $\alpha = (\alpha', 2)$ then

$$D^\alpha u^k = D^{\alpha'} (u^k_{x_n})_{x_n}$$

$$= D^{\alpha'} \left(\sum_{j=1}^{n-1} \sum_{l=1}^m b_j^{kl} u^l_{x_j} + c^k \right)_{x_n} \quad \{ \text{by eqⁿ (ii')} \}$$

$$= D^{\alpha'} \left(\sum_{j=1}^{n-1} \sum_{l=1}^m \left(b_j^{kl} u^l_{x_j x_n} + \sum_{p=1}^m b_j^{kl} z_p u^p_{x_n} u^l_{x_j} \right) \right. \\ \left. + \sum_{p=1}^m (c_{z_p}^k u^p_{x_n}) \right)$$

Thus,

$$D^\alpha u^k(0) = D^{\alpha'} \left(\sum_{j=1}^{n-1} \sum_{l=1}^m b_j^{kl} u^l_{x_j x_n} + \sum_{p=1}^m c_{z_p}^k u^p_{x_n} \right) \Big|_{x=0}$$

The expression on the RHS can be worked out to be a polynomial with non-negative coefficients involving various derivatives of $\{B_j\}_{j=1}^{n-1}$ and C and the derivatives $D^\beta u$, where $|\beta| \leq 1$.

for each multiindices α and each $k = \{1, 2, \dots, n\}$

we compute

$$D^\alpha u^k(0) = P_\alpha^k \left(\dots, D_2^r D_1^s B_j, \dots, D_2^r D_1^s C, \dots, D^\beta u, \dots \right)_{\substack{\alpha=0 \\ u=0}}$$

where P^k denotes some polynomial with nonnegative coefficients.

we deduce for each α, k ,

$$u_\alpha^k = q_\alpha^k \left(\dots, B_{j,r,s}, \dots, C_{r,s}, \dots, u_\beta, \dots \right) \dots (viii)$$

where q_α^k is a polynomial with nonnegative coefficients and $|\beta| \leq |\alpha| - 1$ for each multiindex β on the RHS of eqⁿ (viii).

NOTE: The Cauchy - Kovalévskaya Theorem is valid also for fully nonlinear analytic PDE.